# DISTRIBUTION OF PRESENT VALUE UNDER DIFFERENT PROCESSES FOR FUTURE PAYOFFS

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# DISTRIBUTION OF PRESENT VALUE UNDER DIFFERENT PROCESSES FOR FUTURE PAYOFFS

**Amado Peiró***

**Abstract**

This paper presents the distribution of the present value of a project or asset under different processes for future payoffs. For all hypothesized processes, the coefficient of variation (relative uncertainty) is proportional to the coefficient of variation of the next payoff, the factor of proportionality being function of the discount rate. The variability of the present value may increase or decrease with respect to the discount rate, depending on the type of process. It is higher when payoffs follow a random walk than for the hypothesized processes with independent payoffs, and becomes extremely large for low discount rates.

*Key words*: Coefficient of variation, discounting, present value.
*JEL Classification*: D80, G12.

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## 1. Introduction

Present value relationship, one of the main economic relationships, constitutes the core of many economic theories and models. It is ubiquitous, and is used throughout the valuation of economic projects and a variety of assets, such as companies, stocks, real state and even human lives. The origin of this relationship is very remote, tracing back to the beginning of the 13th century (see Goetzmann, 2004).

This relationship equals the value of the project or asset at a certain moment in time, $t$, to the total of discounted future net payoffs:

$$P_t = \sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i} \qquad (1.1)$$

where $P_t$ is the net present value in $t$, $D_{t+1}, D_{t+2}, \dots$ are net payoffs corresponding to moments $t+1, t+2, \dots$ and $r > 0$ is the discount rate.

Though future payoffs are often unsure, their uncertainty is hardly ever taken into account and the contributions that explicitly deal with the random nature of the present value relationship are very limited Very often only their expected values are used and, consequently, only the expectation of the present value of these payoffs is obtained. One clear example may be found in investment analysis or capital budgeting. As stated in Lee and Tai (2013), three alternative methods are used: statistical, decision-tree and simulation methods. These methods do not allow obtaining the statistical distribution of the present value; even the model proposed by Hillier (1963) presents an ad-hoc treatment of uncertainty of future payoffs (see also Chen and Moore (1982) or Rajaratnam et al. (2014) for a recent application of certainty equivalent method). However, given their uncertain nature, they may be represented by random variables with certain characteristics. Then, if the discount rate is considered deterministic, (1.1) expresses the present value as a combination of random variables with decreasing weights. This paper aims to study the distribution of $P_t$ under different stochastic models for future payoffs. Two features of the distribution of $P_t$ will be specially interesting: the expectation and the variance (or, equivalently, the standard deviation or the coefficient of variation). They will provide measures of the expected value and the risk, respectively. In addition, it will also be interesting to analyze the relationship between these two features and the discount rate, $r$. To achieve these objectives, Section 1 proposes different statistical distributions for future payoffs, Section 2 obtains the distributions of $P_t$ under these distributions, and Section 3 comments on their properties.

## 2. Different models for future payoffs

Future payoffs may follow very different random distributions. While some distributions are rather unrealistic for most projects or assets, others could be reasonably hypothesized in certain situations. In what follows, the presumptions made will be based on the plausibility and mathematical tractability of the distributions. A first convenient assumption will be their normality, as the linear combination of normally distributed payoffs will yield a distribution also normal for $P_t$.[1] These normal distributions are determined by their expectations and variances. Initially, expectations will be considered constant over time for the different payoffs, while variances will be constant or increasing over time, thus reflecting the fact that more distant payoffs are usually more uncertain. These assumptions permit us to consider the following class of models for future payoffs:

$$D_{t+i} \sim N(\overline{D},\ i^{\alpha}\sigma^2),\ i = 1, 2, \ldots \tag{2.1}$$

with $D_{t+i}$ and $D_{t+j}$ being independent random variables if $i \neq j$, $\overline{D} > 0$ and $\alpha \geq 0$. The parameter $\alpha$ regulates the variability of future payoffs. Taking $\alpha = 0,\ \alpha = 1$ and $\alpha = 2,$ the following respective processes are hypothesized:

Process 1 (P1): $D_{t+i} \sim N(\overline{D},\ \sigma^2),\ i = 1, 2, \ldots$

Process 2 (P2): $D_{t+i} \sim N(\overline{D},\ i\sigma^2),\ i = 1, 2, \ldots$

Process 3 (P3): $D_{t+i} \sim N(\overline{D},\ i^2\sigma^2),\ i = 1, 2, \ldots$

[1] In any case, central limit theorems may be invoked when other distributions are considered for future payoffs.

P1 follows from (2.1) with $\alpha = 0$. The uncertainty of future payoffs (measured by their standard deviations or variances) will be constant. Uncertainty does not increase with the time interval to the moment of payoff. When $\alpha = 1$ in (2.1), one obtains P2. In this model, uncertainty increases with the time interval to the payoff; the variances increase proportionally, and the standard deviations increase less than proportionally. That is, the variance of payoff in $t + n$ is $n$ times the variance of payoff in $t + 1$, and the standard deviation of payoff in $t + n$ is $\sqrt{n}$ times the standard deviation of payoff in $t + 1$ for any $n > 1$. P3 follows from (2.1) with $\alpha = 2$. In this case, the standard deviation is proportional to the time interval (a payoff in $t + n$ has a standard deviation equal to $n$ times the standard deviation of payoff in $t + 1$). In all three cases the uncertainty of the next payoff is the same, but the uncertainty of payoffs that will take place in $t + n$, for any $n > 1$, will increase with $\alpha$.

The independence assumption between the different payoffs may be relaxed by supposing that they follow a random walk,

Process 4 (P4): $D_{t+i} = D_{t+i-1} + \varepsilon_{t+i},\ i = 1, 2, \ldots$

with $D_t = \bar{D} > 0,\ \varepsilon_{t+i} \sim N(0,\ \sigma^2)$ for all $i$, and $Cov(\varepsilon_{t+i},\ \varepsilon_{t+j}) = 0$ for any $i, j > 0,\ i \neq j$. In this model the expected value of every future payoff will be constant, $\bar{D}$. As $Var(D_{t+i}) = i\sigma^2$, their variances will increase with the time horizon to the payoff date, and will be equal to those under P2 (proportional to the time interval). This process P4 may also be extended by allowing the expectations of future payoffs to grow at rate $g$:

Process 5 (P5): $D_{t+i} = (1 + g)D_{t+i-1} + \varepsilon_{t+i},\ i = 1, 2, \ldots$

with $D_t = \bar{D}/(1 + g) > 0$, $0 < g < r$, $\varepsilon_{t+i} \sim N(0,\ \sigma^2)$ for all $i$, and $Cov(\varepsilon_{t+i},\ \varepsilon_{t+j}) = 0$ for any $i, j > 0,\ i \neq j$. In this model $g$ is the non-random rate of growth of the expectations of future payoffs. If $g = 0$, P4 would be obtained. As $g > 0$, both the expected value of payoffs and their uncertainty will increase with the time horizon. The expectation and the standard deviation of the first payoff, $D_{t+1}$, are the same for P5 as for the other processes P1-P4. Table 1 shows the standard deviations of different payoffs under P1-P4, and under P5 for different values of $g$.

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=10$ | $i=20$ | $i=50$ |
|---|---|---|---|---|---|---|---|---|
| P1 | 1 | 1 1 1 | | | 1 | 1 | 1 | 1 |
| P2 | 1 | 1.41 | 1.73 | 2 2.24 | | 3.16 | 4.47 | 7.07 |
| P3 | 1 | 2 3 4 | | | 5 | 10 | 20 | 50 |
| P4 | 1 | 1.41 | 1.73 | 2 2.24 | | 3.16 | 4.47 | 7.07 |
| P5 ($g=3\%$) | 1 | 1.44 | 1.79 | 2.09 | 2.38 | 3.64 | 6.09 | 17.30 |
| P5 ($g=10\%$) | 1 | 1.49 | 1.92 | 2.33 | 2.75 | 5.22 | 14.52 | 256.16 |

**Table 1**. Standard deviations of several future payoffs $D_{t+i}$, expressed as times the standard deviation of payoff $D_{t+1}$ $(\sigma)$, under processes P1-P5.

## 3. Distributions of present value under the different models for future payoffs

Under P1-P5, $P_t$ is normally distributed, given that future payoffs are normally distributed. These normal distributions are fully determined by their respective expectations and variances, which are stated in the following propositions.

**3.1.** Under P1,

$$P_t \sim N\left(\frac{\overline{D}}{r}, \frac{\sigma^2}{(1+r)^2-1}\right). \tag{3.1}$$

(See Appendix A1).

**3.2.** Under P2,

$$P_t \sim N\left(\frac{\overline{D}}{r}, \frac{(1+r)^2}{((1+r)^2-1)^2}\sigma^2\right). \tag{3.2}$$

(See Appendix A2).

**3.3.** Under P3,

$$P_t \sim N\left(\frac{\overline{D}}{r}, \frac{(1+r)^4+(1+r)^2}{((1+r)^2-1)^3}\sigma^2\right). \tag{3.3}$$

(See Appendix A3).

**3.4.** Under P4,

$$P_t \sim N\left(\frac{\bar{D}}{r}, \frac{(1+r)^2\sigma^2}{((1+r)^2-1)^2} + \frac{2\sigma^2}{r((1+r)^2-1)} + \frac{2\sigma^2}{r((1+r)^2-1)^2}\right). \tag{3.4}$$

(See Appendix A4).

**3.5.** Under P5,

$$\begin{aligned} P_t \sim N\Bigg(\frac{\bar{D}}{r-g}, \ & \frac{(1+r)^2\sigma^2}{((1+r)^2-1)((1+r)^2-(1+g)^2)} + \\ & + \frac{2(1+g)^3\sigma^2}{(r-g)((1+g)^2-1)((1+r)^2-(1+g)^2)} - \\ & - \frac{2(1+g)\sigma^2}{(r-g)((1+g)^2-1)((1+r)^2-1)}\Bigg). \end{aligned} \tag{3.5}$$

(See Appendix A5).

## 4. Properties of the distributions of present value

Under P1-P4, it is clear that $E(P_t)$ is decreasing and is strictly convex with respect to $r$. $E(P_t)$ tends to $+\infty$ $(0)$ when $r$ tends to $0$ $(+\infty)$. See Fig. 1. Changes in $r$ provoke higher absolute changes in $E(P_t)$ when $r$ is low than when $r$ is high. This does not apply to relative changes; in fact, a relative change in $r$ equal to $\lambda$ implies a relative change in $E(P_t)$ equal to $-\lambda/(1+\lambda)$ for any *r*. Under P5 the expectation of $P_t$ is:

$$E(P_t) = \frac{\bar{D}}{r-g}$$

an expression known as 'Gordon growth model'. When $g$ is close to $r$, $E(P_t)$ will be high and, furthermore, changes in any of these variables will have strong effects on the expectations of $P_t$.

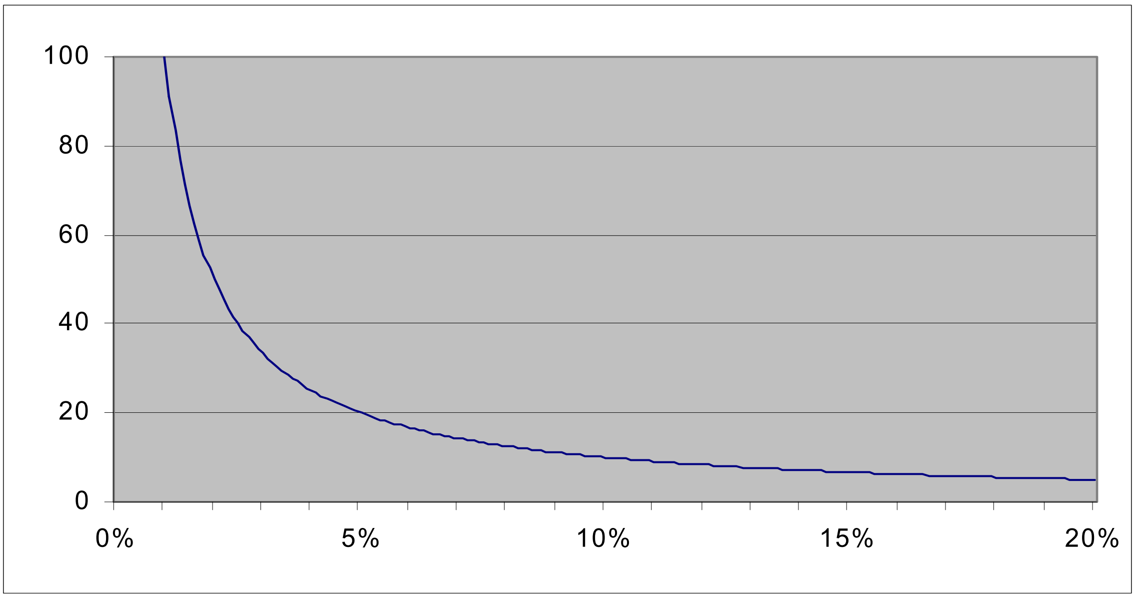


**Fig. 1.** Expectation of the present value $E(P_t)$, expressed as times the expectation of the next payoff $(\overline{D})$, against the discount rate $r$ under processes P1-P4.

A relative measure of risk or uncertainty of $P_t$ will be its coefficient of variation, its standard deviation divided by its expectation. Under P1-P4 the expectations of $P_t$ are directly proportional to $\overline{D}$, and their standard deviations are directly proportional to $\sigma$. Given these proportionality relations, the coefficients of variation of $P_t$ will be directly proportional to the coefficient of variation of the next payoff, $\sigma/\overline{D}$, and then may be written as:

$$CV_i = \frac{\sigma}{\overline{D}} f_i(r),\ \ i = 1, 2, 3 \text{ and } 4 \qquad (4.2)$$

where $CV_1, CV_2, CV_3$ and $CV_4$ denote the coefficients of variation of $P_t$ under processes P1, P2, P3 and P4, respectively, and $f_i(r)$, with $i = 1, 2, 3$ and $4$, denote different functions with $r$ as the only argument.

Table 2 and Fig. 2 show the coefficients of variation as functions of $r$ expressed as times the ratio $\sigma/\overline{D}$. The selected range for $r$ intends to cover a range of plausible values (from 0% to 20%), though the disparity in values for the discount rate in previous research is enormous (see Frederick et al., 2002). Under the different processes, the coefficients of variation present the following properties:

i) Under P1 the coefficient of variation of $P_t$ increases with respect to $r$. However, it is relatively low, ranging approximately from 0 to 0.3 times the ratio $\sigma/\overline{D}$ for $0 < r < 20\%$. Thus, taking, for example, $\sigma/\overline{D} = 0.1$, a ratio which could be a reasonable value in certain circumstances, the coefficient of variation moves approximately from 0 to 3% when $r$ changes from 0 to 20%.

ii) Under P2 the coefficient of variation of $P_t$ is almost constant, ranging approximately from 0.50 to 0.55 times the ratio $\sigma/\overline{D}$ for $0 < r < 20\%$. This implies that, for example, for $\sigma/\overline{D} = 0.1$ the coefficient of variation moves approximately from 5.0% to 5.5% when $r$ changes from 0 to 20%.

iii) Under P3 the coefficient of variation of $P_t$ is much higher than under P1 or P2, and decreases with respect to $r$. The diminution is important. For example, for $r = 1\%$ the coefficient of variation is approximately 5 times the ratio $\sigma/\overline{D}$, while for $r = 5\%$ it is about 2.3 times. As reflected in Fig. 2, when $r$ is high, changes in $r$ have a small effect on the coefficient of variation of $P_t$, but when $r$ is low, they have a large effect.

iv) Under P4 the coefficient of variation of $P_t$ is higher than under P1-P3. Its behavior is similar to that found under P3. It also decreases with respect to $r$, and the decrement is very prominent for low values of $r$. When $r$ is high, changes in $r$ have a much more limited effect on the coefficient of variation of $P_t$.

| | $r = 1\%$ | $r = 3\%$ | $r = 5\%$ | $r = 10\%$ | $r = 15\%$ | $r = 20\%$ |
|---|---|---|---|---|---|---|
| P1 | 0.07 | 0.12 | 0.16 | 0.22 | 0.26 | 0.30 |
| P2 | 0.50 | 0.51 | 0.51 | 0.52 | 0.53 | 0.55 |
| P3 | 5.04 | 2.95 | 2.32 | 1.70 | 1.44 | 1.28 |
| P4 | 7.12 | 4.17 | 3.28 | 2.40 | 2.03 | 1.81 |

**Table 2**. Coefficients of variation of the present value, expressed as times the coefficient of variation of the next payoff, $\sigma/\overline{D}$, for different values of the discount rate $r$ under P1-P4.

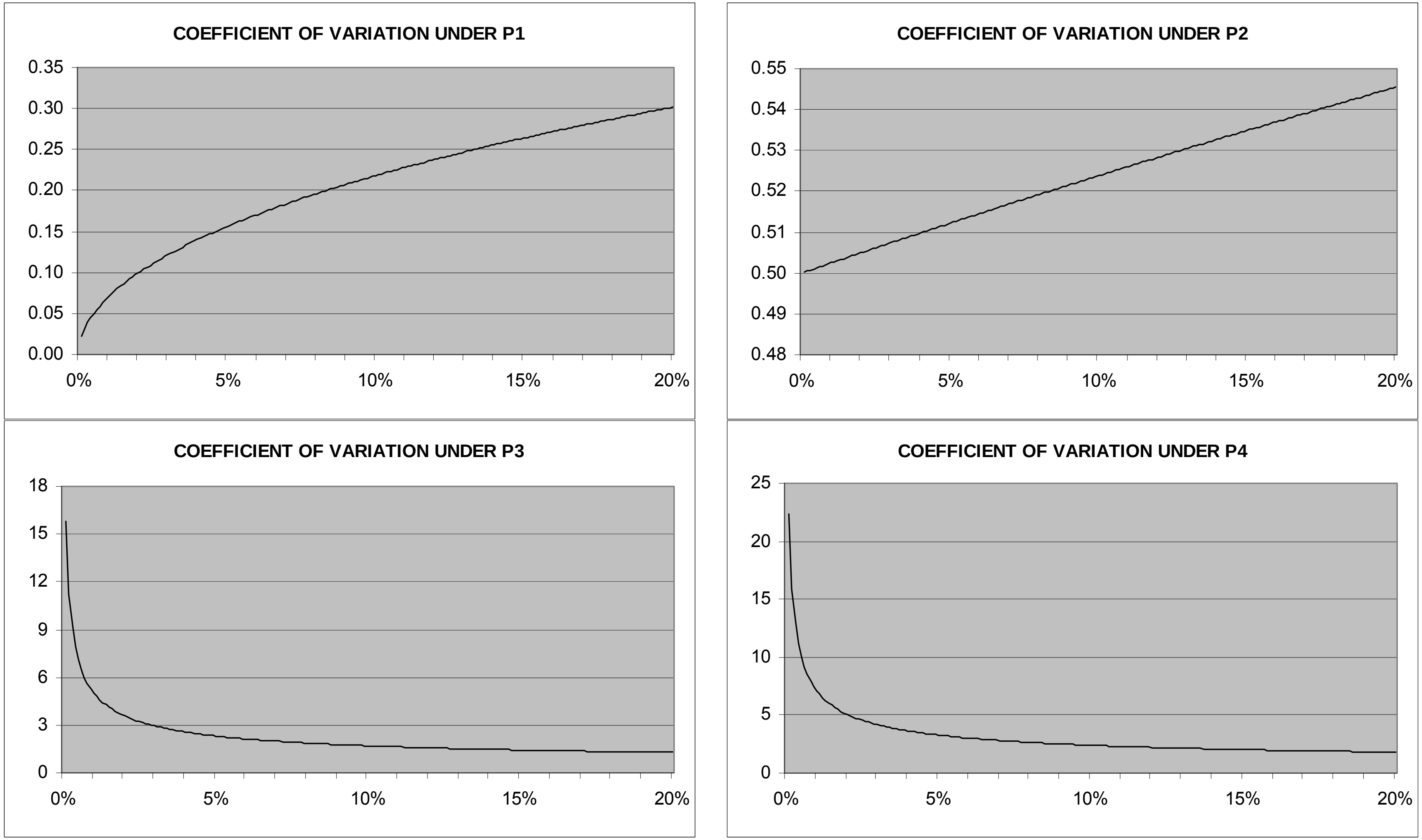


**Fig. 2.** Coefficients of variation of the present value, expressed as times the coefficient of variation of the next payoff $(\sigma/\bar{D})$, against the discount rate $r$ under processes P1-P4.

The standard deviation of future independent payoffs, $D_{t+i}$, with $i > 1$, is greater under P3 than under P2, and greater under P2 than under P1; therefore, for a given $r$, $CV_1 < CV_2 < CV_3$. It should be noted that the relative difference depends on $r$. Thus, for example, for $r = 20\%$, the coefficient of variation under P3 is a little more than double the coefficient under P2, but for $r = 1\%$, it is more than tenfold. It is also interesting to compare $CV_2$ and $CV_4$. The expectations and standard deviations of future payoffs are exactly equal under these models. However, the variance of $P_t$ under P2 is:

$$\frac{(1+r)^2\sigma^2}{((1+r)^2-1)^2} \tag{4.3}$$

but under P4 it is

$$\frac{(1+r)^2\sigma^2}{((1+r)^2-1)^2}+\frac{2\sigma^2}{r((1+r)^2-1)}+\frac{2\sigma^2}{r((1+r)^2-1)^2} \tag{4.4}$$

One can observe that (4.3) is the first summand of (4.4). It is the part of the variance of $P_t$ that is due to the variances of future payoffs. The difference between (4.4) and (4.3),

$$\frac{2\sigma^2}{r((1+r)^2-1)}+\frac{2\sigma^2}{r((1+r)^2-1)^2}$$

is the component of the variance of $P_t$ under P4 due to the co-variability of future payoffs. As this second component is positive, the variance and, therefore, the coefficient of variation of $P_t$ will always be higher under P4 than under P2. In fact, for typical values of $r$, the second component due to the co-variability is much larger than the first component due to the variability, and therefore $CV_4$ is much higher than $CV_2$, as shown in Fig. 2 for values of $r$ comprised between 0% and 20%,

Let us now examine the coefficient of variation of the present value under P5, $CV_5$. As both the expectation and the standard deviation of the present value depend on both $r$ and $g$, one could think that $CV_5$ also depends on both $r$ and $g$. However, the following proposition states that it only depends on $r$, and that it is equal to that under P4.

**Proposition 4.1** Under P5, the coefficient of variation of the present value is:

$$\frac{r}{\overline{\overline{D}}}\sqrt{\frac{(1+r)^2\sigma^2}{((1+r)^2-1)^2}+\frac{2\sigma^2}{r((1+r)^2-1)}+\frac{2\sigma^2}{r((1+r)^2-1)^2}}$$

***Proof.*** It is easy (but rather cumbersome) to prove that the quotient of the variance and the square of the expectation of the present value under P5 is equal to that under P4. Thus, equality of the coefficients of variation follows.

Therefore, under P5, $g$ determines the expectations and the standard deviations of future payoffs. It also determines the expectation and the standard deviation of the present value. However, it does not determine the coefficient of variation of the present value, which is equal to that under P4, $CV_5 = CV_4$.

## 5 Conclusions

This paper provides the distribution of the present value under different feasible processes for future payoffs, with special attention paid to its relative variability measured by the coefficient of variation. The results obtained with the hypothesized processes show that the relationship of the variability of the present value with the discount rate depends on the process of the future payoffs taken into account. While processes with independent payoffs of constant or slowly increasing uncertainty may present an increasing (though limited) variability of the present value with respect to the discount rate, the opposite pattern may occur when the uncertainty of the payoffs increases more quickly. Processes with dependent payoffs present a higher variability of the present value due to the co-variation of future payoffs, and this clearly decreases with respect to the discount rate. When considering extensions of a random walk for future payoffs that also allow expectations to increase over time at a given rate, it is found that the coefficient of variation of the present value does not depend on the rate of growth.

**Appendix A1**

$$E\left(P_t\right)=E\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-i} E\left(D_{t+i}\right)=\bar{D}\sum_{i=1}^{\infty}(1+r)^{-i}=\frac{\bar{D}}{r}.$$

$$Var\left(P_t\right)=Var\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-2i} Var\left(D_{t+i}\right)=\sigma^2\sum_{i=1}^{\infty}(1+r)^{-2i}=\frac{\sigma^2}{(1+r)^2-1}.$$

**Appendix A2**

$$E\left(P_t\right)=E\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-i} E\left(D_{t+i}\right)=\bar{D}\sum_{i=1}^{\infty}(1+r)^{-i}=\frac{\bar{D}}{r}.$$

$$Var\left(P_t\right)=Var\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-2i} Var\left(D_{t+i}\right)=\sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i}.$$

$\sum_{i=1}^{\infty}i(1+r)^{-2i}$ is an arithmetic-geometric series and then:

$$\sum_{i=1}^{\infty}i(1+r)^{-2i}=\frac{(1+r)^{-2}}{1-(1+r)^{-2}}+\frac{(1+r)^{-4}}{\left(1-(1+r)^{-2}\right)^2}=\frac{(1+r)^2}{\left((1+r)^2-1\right)^2}.$$

Therefore,

$$Var\left(P_t\right)=\frac{(1+r)^2}{\left((1+r)^2-1\right)^2}\sigma^2.$$

**Appendix A3**

$$E\left(P_t\right)=E\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-i} E\left(D_{t+i}\right)=\bar{D}\sum_{i=1}^{\infty}(1+r)^{-i}=\frac{\bar{D}}{r}.$$

$$Var\left(P_t\right)=Var\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-2i} Var\left(D_{t+i}\right)=\sigma^2\sum_{i=1}^{\infty}i^2(1+r)^{-2i}.$$

The series $\sum_{i=1}^{\infty}i^2(1+r)^{-2i}$ is convergent (D'Alembert criterion). Then, if

$$S_i=(1+r)^{-2}+4(1+r)^{-4}+9(1+r)^{-6}+\cdots+i^2(1+r)^{-2i},$$

$$\frac{S_i}{(1+r)^2}=(1+r)^{-4}+4(1+r)^{-6}+9(1+r)^{-8}+\cdots+i^2(1+r)^{-2i-2},$$

$$\frac{(1+r)^2-1}{(1+r)^2}S_i=(1+r)^{-2}+3(1+r)^{-4}+5(1+r)^{-6}+\cdots+(2i-1)(1+r)^{-2i}-i^2(1+r)^{-2i-2},$$

$$\frac{\left((1+r)^2-1\right)}{(1+r)^4}S_i=(1+r)^{-4}+3(1+r)^{-6}+5(1+r)^{-6}+\cdots+(2i-1)(1+r)^{-2i-2}-i^2(1+r)^{-2i-4},$$

$$\frac{(1+r)^2-1}{(1+r)^2}S_i-\frac{(1+r)^2-1}{(1+r)^4}S_i=\left(\frac{(1+r)^2-1}{(1+r)^2}\right)^2 S_i=(1+r)^{-2}+2(1+r)^{-4}+2(1+r)^{-6}+\cdots+$$

$$+\cdots+2(1+r)^{-2i+2}+2(1+r)^{-2i}+\left(i^2-2i+1\right)(1+r)^{-2i-2}+i^2(1+r)^{-2i-4}=$$

$$=(1+r)^{-2}+\frac{2(1+r)^{-2i-2}-2(1+r)^{-4}}{(1+r)^{-2}-1}+(i-1)^2(1+r)^{-2i-2}+i^2(1+r)^{-2i-4}.$$

As $\sum_{i=1}^{\infty} i^2(1+r)^{-2i-4}$ and $\sum_{i=1}^{\infty}(i-1)^2(1+r)^{-2i-2}$ are convergent given that

$\lim_{i\to\infty}\frac{(i+1)^2(1+r)^{-2i-6}}{i^2(1+r)^{-2i-4}}=\lim_{i\to\infty}\frac{(i+1)^2}{i^2(1+r)^2}<1$, and

$\lim_{i\to\infty}\frac{i^2(1+r)^{-2n-4}}{(i-1)^2(1+r)^{-2n-2}}=\lim_{i\to\infty}\frac{i^2}{(i-1)^2(1+r)^2}<1$, then,

$$\sum_{i=1}^{\infty} i^2(1+r)^{-2i}=\lim_{i\to\infty}S_i=\left(\frac{(1+r)^2}{(1+r)^2-1}\right)^2\lim_{i\to\infty}\left(\frac{1}{(1+r)^2}+\frac{2(1+r)^{-2i-2}-2(1+r)^{-4}}{(1+r)^{-2}-1}\right)=$$

$$=\left(\frac{(1+r)^2}{(1+r)^2-1}\right)^2\left(\frac{1}{(1+r)^2}+\frac{2}{(1+r)^4-(1+r)^2}\right)=\frac{(1+r)^4+(1+r)^2}{\left((1+r)^2-1\right)^3},$$

and $Var(P_t)=\frac{(1+r)^4+(1+r)^2}{\left((1+r)^2-1\right)^3}\sigma^2$.

**Appendix A4**

$$E(P_t)=E\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-i}E(D_{t+i})=\bar{D}\sum_{i=1}^{\infty}(1+r)^{-i}=\frac{\bar{D}}{r}.$$

$$Var(P_t)=Var\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-2i}Var(D_{t+i})+2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j}Cov\left(D_{t+i},D_{t+j}\right)=$$

$$=\sum_{i=1}^{\infty}(1+r)^{-2i}\sum_{j=1}^{i}\sigma^2+2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j}\sum_{k=1}^{i}\sigma^2=$$

$$=\sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i}+2\sigma^2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}i(1+r)^{-i}(1+r)^{-j}.$$

$Var(P_t)$ is expressed as the sum of two components: the first, $\sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i}$, is the part of $Var(P_t)$ due to the variability of future payoffs, while the second, $2\sigma^2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}i(1+r)^{-i}(1+r)^{-j}$, is the part of $Var(P_t)$ due to the co-variability of future different payoffs. The first component is:

$\sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i}=\frac{(1+r)^2}{\left((1+r)^2-1\right)^2}\sigma^2$ (see Appendix A2), and the second is

$$2\sigma^2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}i(1+r)^{-i}(1+r)^{-j}=$$

$$=2\sigma^2\left((1+r)^{-1}\sum_{j=2}^{\infty}(1+r)^{-j}+2(1+r)^{-2}\sum_{j=3}^{\infty}(1+r)^{-j}+3(1+r)^{-3}\sum_{j=4}^{\infty}(1+r)^{-j}+\cdots\right)=$$

$$=2\sigma^2\left(\frac{1}{r(1+r)^2}+\frac{2}{r(1+r)^4}+\frac{3}{r(1+r)^6}+\cdots\right)=\frac{2\sigma^2}{r\left((1+r)^2-1\right)}+\frac{2\sigma^2}{r\left((1+r)^2-1\right)^2},$$

given that $\frac{1}{r(1+r)^2}+\frac{2}{r(1+r)^4}+\frac{3}{r(1+r)^6}+\cdots$ is an arithmetic-geometric series with first term equal to $\frac{1}{r(1+r)^2}$, difference equal to $\frac{1}{r(1+r)^2}$, and ratio equal to $\frac{1}{(1+r)^2}$.

Then, $Var(P_t)=\frac{(1+r)^2\sigma^2}{\left((1+r)^2-1\right)^2}+\frac{2\sigma^2}{r\left((1+r)^2-1\right)}+\frac{2\sigma^2}{r\left((1+r)^2-1\right)^2}$.

**Appendix A5**

$$E\left(P_t\right)=E\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-i}E\left(D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-i}(1+g)^{i}D_t=\frac{\bar{D}}{r-g}$$

$$Var\left(P_t\right)=Var\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-2i}Var\left(D_{t+i}\right)+2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j}Cov\left(D_{t+i},D_{t+j}\right).$$

$Var(P_t)$ is expressed as the sum of two components: the first, $\sum_{i=1}^{\infty}(1+r)^{-2i}Var(D_{t+i})$, is the part of $Var(P_t)$ due to the variability of future payoffs, while the second, $2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j}Cov(D_{t+i},D_{t+j})$, is the part of $Var(P_t)$ due to the co-variability of future different payoffs. The first component is:

$$\sum_{i=1}^{\infty}(1+r)^{-2i}Var\left(D_{t+i}\right)=\sum_{i=1}^{\infty}(1+r)^{-2i}\sum_{j=1}^{i}(1+g)^{2(j-1)}\sigma^2=$$

$$=\sigma^2\left[(1+r)^{-2}+\left((1+r)^{-4}\left(1+(1+g)^2\right)\right)+\left((1+r)^{-6}\left(1+(1+g)^2+(1+g)^4\right)\right)+\cdots\right]=$$

$$=\sigma^2\left[\sum_{i=1}^{\infty}(1+r)^{-2i}+(1+g)^2\sum_{i=2}^{\infty}(1+r)^{-2i}+(1+g)^4\sum_{i=3}^{\infty}(1+r)^{-2i}+\cdots\right]=$$

$$=\sigma^2\left[\frac{(1+r)^{-2}}{1-(1+r)^{-2}}+(1+g)^2\frac{(1+r)^{-4}}{1-(1+r)^{-2}}+(1+g)^4\frac{(1+r)^{-6}}{1-(1+r)^{-2}}+\cdots\right]=\sigma^2\frac{\dfrac{(1+r)^{-2}}{1-(1+r)^{-2}}}{1-(1+g)^2(1+r)^{-2}}=$$

$$=\frac{\sigma^2}{\left((1+r)^2-1\right)\left(1-(1+g)^2(1+r)^{-2}\right)}=\frac{(1+r)^2\sigma^2}{\left((1+r)^2-1\right)\left((1+r)^2-(1+g)^2\right)}.$$

And the second component is:

$$2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j}Cov\left(D_{t+i},D_{t+j}\right)=$$

$$=2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j}E\left[\left(\sum_{k=1}^{i}(1+g)^{i-k}\varepsilon_{t+k}\right)\left(\sum_{l=1}^{j}(1+g)^{j-l}\varepsilon_{t+l}\right)\right]=$$

$$=2\sigma^2\begin{bmatrix}(1+r)^{-1}\left((1+r)^{-2}(1+g)+(1+r)^{-3}(1+g)^2+(1+r)^{-4}(1+g)^3+\ldots\right)+\\(1+r)^{-2}\left((1+r)^{-3}\left((1+g)^3+(1+g)\right)+(1+r)^{-4}\left((1+g)^4+(1+g)^2\right)+\cdots\right)+\\(1+r)^{-3}\left((1+r)^{-4}\left((1+g)^5+(1+g)^3+(1+g)\right)+(1+r)^{-5}\left((1+g)^6+(1+g)^4+(1+g)^2\right)+\cdots\right)+\\+\cdots\end{bmatrix}=$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} (1+r)^{-i} (1+r)^{-j} \sum_{k=1}^{i} (1+g)^{j-i+2(k-1)} =$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} (1+r)^{-i} (1+r)^{-j} \frac{(1+g)^{j-i} - (1+g)^{j+i}}{1-(1+g)^2} =$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} (1+r)^{-i} \left( (1+r)^{-i-1} \frac{(1+g)-(1+g)^{2i+1}}{1-(1+g)^2} + (1+r)^{-i-2} \frac{(1+g)^2-(1+g)^{2i+2}}{1-(1+g)^2} + \cdots \right) =$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} (1+r)^{-i} \left( \frac{(1+r)^{-i-1} \frac{(1+g)-(1+g)^{2i+1}}{1-(1+g)^2}}{1-(1+r)^{-1}(1+g)} \right) = 2\sigma^2 \sum_{i=1}^{\infty} \frac{(1+r)^{-2i-1}\left((1+g)-(1+g)^{2i+1}\right)}{\left(1-(1+g)^2\right)\left(1-(1+r)^{-1}(1+g)\right)} =$$

$$= 2\sigma^2 \left( \sum_{i=1}^{\infty} \frac{(1+r)^{-2i-1}(1+g)}{\left(1-(1+g)^2\right)\left(1-(1+r)^{-1}(1+g)\right)} - \sum_{i=1}^{\infty} \frac{(1+r)^{-2i-1}(1+g)^{2i+1}}{\left(1-(1+g)^2\right)\left(1-(1+r)^{-1}(1+g)\right)} \right) =$$

$$= 2\sigma^2 \left( \frac{\frac{(1+\mathrm{r})^{-3}(1+g)}{\left(1-(1+g)^2\right)\left(1-(1+r)^{-1}(1+g)\right)}}{1-(1+r)^{-2}} - \frac{\frac{(1+\mathrm{r})^{-3}(1+g)^3}{\left(1-(1+g)^2\right)\left(1-(1+r)^{-1}(1+g)\right)}}{1-(1+r)^{-2}(1+g)^2} \right) =$$

$$= 2\sigma^2 \left( \begin{array}{l} \frac{(1+\mathrm{r})^{-3}(1+g)}{\left(1-(1+g)^2\right)\left(1-(1+r)^{-1}(1+g)\right)\left(1-(1+r)^{-2}\right)} - \\ - \frac{(1+\mathrm{r})^{-3}(1+g)^3}{\left(1-(1+g)^2\right)\left(1-(1+r)^{-1}(1+g)\right)\left(1-(1+r)^{-2}(1+g)^2\right)} \end{array} \right) =$$

$$= 2\sigma^2 \left( \frac{(1+g)}{\left((1+g)^2-1\right)(g-r)\left((1+r)^2-1\right)} - \frac{(1+g)^3}{\left((1+g)^2-1\right)(g-r)\left((1+r)^2-(1+g)^2\right)} \right).$$

Then,

$$Var\left(P_t \middle| \Omega_t\right) = \frac{(1+r)^2 \sigma^2}{\left((1+r)^2-1\right)\left((1+r)^2-(1+g)^2\right)} +$$

$$+ \left( \frac{(1+g)^3}{\left((1+g)^2-1\right)(r-g)\left((1+r)^2-(1+g)^2\right)} - \frac{(1+g)}{\left((1+g)^2-1\right)(r-g)\left((1+r)^2-1\right)} \right) 2\sigma^2 .$$

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