TESTING FOR THE GENERAL FRACTIONAL UNIT ROOT HYPOTHESIS IN THE TIME DOMAIN

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Testing for the General Fractional Unit Root Hypothesis in the Time Domain *

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Abstract

In this paper we propose a Lagrange Multiplier test as well as a family of asymptotically equivalent LS-based testing procedures which are intended to detect general forms of fractional integration at the long-run and/or the cyclical component of a time series. Our setting extends Robinson´s (1994) approach to the time domain and generalizes the procedures in Agiakloglou and Newbold (1994), Tanaka (1999) and Breitung and Hassler (2002) by allowing for single or multiple fractional unit roots at any frequency in $[0, \pi]$. Our testing procedure can be easily implemented in practical settings and is flexible enough to account for a broad family of long- and short-memory specifications, including ARMA-type and/or GARCH-type dynamics, among others. Furthermore, it has power against different types of alternative hypotheses and inference is conducted under critical values drawn from a standard chi-squared distribution, independently of the long-memory parameters.

Keywords: LM tests, nonstationarity, fractional integration, cyclical integration
JEL classification: C20, C22

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1 Introduction

Modelling and forecasting macroeconomic and financial variables is at the forefront of the applied time-series econometric literature. These series are usually characterized by strongly persistent correlation structures over long intervals of time. In this paper, we propose several test statistics to detect general forms of fractional integration in the time domain. Our approach belongs to the Lagrange-multiplier (LM) framework studied in Robinson (1991, 1994), Agiakloglou and Newbold (1994), Tanaka (1999), Breitung and Hassler (2002) and Nielsen (2004, 2005). In particular, we propose standard LM tests for multiple fractional integration, as well as a family of asymptotically equivalent tests in the linear regression model $Y_t = \sum_{s=1}^{n} \phi_s X_{st}(Y_t) + u_t$, where Y_t is directly determined under the null hypothesis and the regressors $X_{st}(Y_t)$ are straightforwardly computed by linearly filtering Y_t. This approach has remarkable methodological advantages. It can be easily implemented for practical settings and is flexible enough to account for a broad family of long- and short-memory specifications. Furthermore, it also has power against different types of alternative hypotheses, and it allows inference to be conducted under critical values which are drawn from a standard chi-squared distribution, independently of the long-memory parameters.

More specifically, the tests we discuss are formally intended to detect general long memory patterns embedded in the autoregressive filter

$$(1-L)^{d_1}\left[\prod_{i=2}^{k+1}(1-2\cos\gamma_i L+L^2)^{d_i}\right](1+L)^{d_{k+2}}$$

where d_i, $i = 1, ..., k+2$, are possibly non-integer values, γ_i, $1 \leq i \leq k+1$, are frequencies in $(0, \pi)$ that characterize the cyclical behavior (periodicity) of the data, and L is the conventional back-shift operator. The filter also allows for long-memory patterns at the zero and Nyquist frequencies. This is the basic data generating process analyzed in Robinson (1994), which is able to capture both long-range dependence and periodic cyclical fluctuations through the convolution of Gegenbauer processes. It generates theoretical autocovariances that decay hyperbolically and sinusoidally, a feature that is manifested in a number of periodic time series. Particular cases of this specification include the well-known fractional unit root model, as well as pure cyclical and seasonal models which are routinely applied to fit both economic and non-economic variables. For instance, cyclical models have been used to explain macroeconomic dynamics by Gray, Zhang and Woodward (1989), Ramachandran and Beaumont (2001), Gil-Alana and Robinson (2001), Gil-Alana (2005), and Smallwood and Norrbin (2006), among many others. Recent studies focusing on non-economic variables have analyzed, for instance, atmospheric levels of CO_2 (Woodward, Cheng and Gray, 1998), wind speed (Bouette *et al.*, 2006), or power demand (Soares and Souza, 2006). The extant literature on seasonal and non-seasonal models embedded in this general framework (both integrated and fractionally integrated) is overwhelming.

Our setting extends Robinson's approach to the time domain and generalizes the procedures in Agiakloglou and Newbold (1994), Tanaka (1999) and Breitung and Hassler (2002) by allowing for single or multiple fractional unit roots at any frequency in $[0, \pi]$. Furthermore, we allow for different types of errors in the data generating process (DGP) which include martingale differences sequences (MDS) and weakly correlated errors, thus allowing for ARMA and/or time varying volatility patterns. As in the frequency-domain case, the tests do not require formal knowledge of the true values of the fractionally-integrated coefficients. These are mainly intended for formally pretesting hypotheses about the extent of cyclical and non-cyclical persistence, and to construct confidence sets that include the true values of the long-memory coefficients with a certain asymptotic coverage level. This is valuable for descriptive inference and, furthermore, provides reliable values for initiating optimization routines upon which several estimation procedures, such as (quasi) maximum likelihood procedures, build on.

The remaining of the paper is organized as follows. Section 2 introduces the general setting and discusses the set of sufficient conditions for the LM tests. Section 3 introduces the standard Lagrange Multiplier test and discusses its asymptotic distribution. Section 4 discusses regression-based tests. The specific form of the regression to be used depending on the type of errors in the DGP, the relevant test statistics, and their asymptotic distributions, is discussed in several theorems. Section 5 analyzes the finite-sample performance of the tests by means of Monte Carlo experimentation. Section 6 summarizes the main conclusions. Finally, the mathematical proofs of the main statements are collected in a technical appendix.

In what follows, '$\Rightarrow$' and '$\xrightarrow{p}$' denote weak convergence and convergence in probability, respectively, as the sample size is allowed to diverge. The variable $\mathbb{I}_{(\cdot)}$ is an indicator function that takes value equal to one if the condition in the subscript is fulfilled and zero otherwise. Finally, vectors and matrices are denoted through bold letters.

2 The general fractionally integrated model

Let $\xi_\gamma(L;\delta)$ be a Gegenbauer polynomial in the lag operator defined as follows,

$$\xi_\gamma(L;\delta) = (1 - 2\cos\gamma L + L^2)^\delta \tag{1}$$

where the long-memory parameter δ can take non-integer values and controls the extent of time dependence. The parameter γ is a so-called Gegenbauer frequency in $[0,\pi]$, and controls the periodicity of the resulting time series.

Define the following generalization of (1), given the set of long-memory parameters $\boldsymbol{\delta} = (\delta_1, ..., \delta_{k+2})'$, $\boldsymbol{\delta} \in R^{k+2}$, and the vector of frequencies $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_{k+2})'$

$$\Delta_{\boldsymbol{\gamma}}(L;\boldsymbol{\delta}) \equiv (1-L)^{\delta_1}\left[\prod_{i=2}^{k+1}\xi_{\gamma_i}(L;\delta_i)\right](1+L)^{\delta_{k+2}} \tag{2}$$

such that $\gamma_1 < \gamma_2 < \cdots < \gamma_{k+2}$, and by definition $\gamma_1 = 0$, $\gamma_{k+2} = \pi$. The resultant filter allows for multiple cyclical components for the $k+1$ seasonal frequencies involved $\{\gamma_s : s > 1\}$, as well as for a long-run trend at the zero frequency, γ_1. For simplicity of notation, the dimension of $\boldsymbol{\delta}$ is denoted as $n \geq 1$. Specification (2) encompasses different types of filters. In addition to the well-known fractionally integrated unit-root model, major examples for empirical purposes include pure cyclical models (which arise by restricting $\boldsymbol{\delta}$), pure seasonal models (which arise by restricting $\boldsymbol{\gamma}$), and any convolution of these. We shall briefly discuss the properties of these restricted models at the end of this section.

We consider that the observable process, $\{x_t,\ t = 1, ..., T\}$, admits the following characterization

$$\Delta_{\boldsymbol{\gamma}}(L;\boldsymbol{\delta})\, x_t = \varepsilon_t \tag{3}$$

where ε_t is a covariance stationary noise process with spectral density that is bounded and bounded away from zero at all frequencies. In the most general case considered in this paper, we will say that x_t is generated by a *General Fractionally Integrated* process of order $\boldsymbol{\delta}$, denoted as $x_t \sim$GFI$(\boldsymbol{\delta})$. The study of particular cases (such as zero frequency, seasonal models, and cyclical models) arises straightforwardly by suitably restricting $\Delta_{\boldsymbol{\gamma}}(L;\boldsymbol{\delta})$. For instance, pure cyclical models arise by restricting $\boldsymbol{\delta}$ and setting the long-memory parameters corresponding to the zero and Nyquist frequencies to zero. The restricted filter is $\prod_{i=1}^{n} \xi_{\lambda_i}(L;\delta_i)$, with dimension $n \geq 1$. When $n = 1$, x_t is said to be generated by a GARMA model, whereas $n > 1$ leads to so-called n-factor GARMA models, which exhibit stationary long-memory patterns if $0 < \delta_i < 1/2$; see Woodward *et al.* (1998), and Ramachandran and Beaumont (2001) for a discussion of the statistical properties of these models. The generalizations (for instance, allowing for stationary short-run dynamics) are able to encompass both ARMA and ARFIMA models as particular cases.

Similarly, pure seasonal models (SARFIMA) arise by restricting both the dimension and the value of $\boldsymbol{\gamma}$ aiming to relate the frequencies to the periodicity of the data, say S. For instance, if S is even, then $\gamma_1 = 0$ and $\gamma_{i+1} = 2\pi i/S$, $i = 1, ..., [S/2]-1$, where the corresponding filter is now given by

$$(1-L)^{\delta_1}\left[\prod_{i=2}^{[S/2]-1} \xi_{\gamma_i}(L;\delta_i)\right](1+L)^{\delta_n} \tag{4}$$

with dimension $n = [S/2]+1$. When S is odd, the component $(1+L)^{\delta_n}$, which corresponds to a cycle of two periods, is simply omitted and the model has $[S/2]$ parameters. A special case is $\delta_1 = ... = \delta_n = 1$, from which the filter $\left(1-L^S\right)$ originating a seasonal random walk arises. By allowing non-integer values in $\boldsymbol{\delta}$, x_t is said to be generated by a seasonal fractionally integrated process of order $\boldsymbol{\delta}$; see, among others, Hassler (1994) and references therein.

Finally, the well-known fractional unit root model of order d, denoted FI(d), arises after removing all the terms related to the non-zero frequencies, *i.e.*, by considering the $(1-L)^d$ filter related to the zero frequency $\gamma = 0$, which corresponds to the ARFIMA(0,d,0) model.

For empirical purposes, the main interest lies in testing whether $\boldsymbol{\delta} = \mathbf{d}$, with $\mathbf{d} \in R^n$ being specified *a priori*, against the alternative for which the order of integration is $\mathbf{d}+\boldsymbol{\theta}$, $\boldsymbol{\theta} \neq \mathbf{0}$. Thus, the hypothesis of interest is generally stated as

$$\text{H}_0 : \boldsymbol{\delta} = \mathbf{d}, \text{ or } \ \text{H}_0 : \boldsymbol{\theta} = \mathbf{0}, \tag{5}$$

against the alternative hypothesis that H_0 is false, *i.e.*, $\text{H}_1 : \boldsymbol{\delta} \neq \mathbf{d}$ or $\text{H}_1 : \boldsymbol{\theta} \neq \mathbf{0}$.

3 Testing procedures

3.1 Preliminaries

We start our theoretical analysis by introducing and discussing the initial set of assumptions and general notational issues which are valid for both the standard LM and the regression-based tests. We also provide several key definitions for this context.

Assumption $\mathcal{A}$:
i) The observable process $\{x_t,\ t = 1, ..., T\}$ *is generated by* $\Delta_{\boldsymbol{\gamma}} (L; \mathbf{d})\, x_t = \varepsilon_t \mathbb{I}_{(t>0)}$, *with* $\Delta_{\boldsymbol{\gamma}} (L; \mathbf{d})$ *defined in* (2), *and* $\mathbf{d}$ *being a possibly non-integer vector in* R^n, $n \geq 1$.
ii) The innovation process $\{\varepsilon_t, \mathcal{G}_t\}_{-\infty}^{\infty}$, $\mathcal{G}_t = \sigma(\varepsilon_j : j \leq t)$, *forms a martingale difference sequence and verifies* $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2 < \infty$, $E(\varepsilon_t^2 | \mathcal{G}_{t-1}) > 0$ *almost surely, with one of the following restrictions holding true:*

ii.a) $\{\varepsilon_t\}$ *is independent and identically distributed and* $E(|\varepsilon_t^4|^{1+r})$ *absolutely bounded for some* $r > 0$.

ii.b) $\{\varepsilon_t\}$ *is strictly stationary and ergodic with*

$$\sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} ... \sum_{l_7=-\infty}^{\infty} |\kappa_\varepsilon (0, l_1, ..., l_7)| < \infty,$$

where $\kappa_\varepsilon (0, l_1, ..., l_7)$ *is the eigth-order joint cumulant of* $\{\varepsilon_t\}$.

Some comments follow. We consider the most general case under the null hypothesis given by $x_t \sim$GFI$(\mathbf{d})$. Simpler specifications (*e.g.*, pure seasonal models) arise considering restricted versions of $\Delta_{\boldsymbol{\gamma}} (L; \mathbf{d})\, x_t$, for which our conclusions extend straightforwardly. Condition *i)* also sets $x_j = \varepsilon_j = 0$ for any $j \leq 0$, so we consider the realizations from a truncated stochastic process. This assumption has become standard in the fractional unit root literature, because it may permit the observable processes to be well-defined in the mean-square sense regardless of the values of $\mathbf{d}$. In the context of the present paper, however, it does not play a major role and the relevant results hold both if we consider $\{\varepsilon_t\}_{t\geq 1}$ or $\{\varepsilon_t\}_{-\infty}^{\infty}$. Condition *ii.a)* can be weakened by requiring that, conditional on the σ-field of events $\mathcal{G}_t$, moments up to the fourth-order (and suitable cross-products of elements of ε_t) equal the corresponding unconditional moments, so that essentially $\{\varepsilon_t\}$ is only required to behave as an i.i.d process up to the fourth-order moment. The main purpose of *ii.b)* is to allow for time-varying conditional volatility patterns in $\{\varepsilon_t\}$. This requires additional restrictions limiting the extent of temporal dependence, which are provided by restricting the absolute summability of the eight-order joint cumulants. This condition is similar to that in Gonçalves and Kilian (2007) and Demetrescu, Kuzin and Hassler (2007). More general errors, allowing for short-run dynamics in mean, are studied later on. Finally, we do not require normality, since this is not essential to derive the asymptotic theory, but we note that efficiency in Gaussian-score based procedures would only be attainable under that restriction.

Before deriving the Lagrange Multiplier type test statistics, we consider the following definitions, which are relevant for notational convenience.

Definition 3.1. *For all* $j \geq 1$ *and* $\gamma \in [0, \pi]$, *define the non-stochastic weighting process* $\omega_j(\gamma)$ *as follows,*

$$\omega_j(\gamma) = \begin{cases} 1/j, & if\ \gamma = 0 \\ 2j^{-1}\cos(j\gamma), & if\ \gamma \in (0, \pi) \\ (-1)^j / j, & if\ \gamma = \pi \end{cases}. \tag{6}$$

Similarly, for $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_n)'$ *such that* $\gamma_s \in [0, \pi]$, $s = 1, ..., n$, *define*

$$\boldsymbol{\omega}_j(\boldsymbol{\gamma}) = (\omega_j(\gamma_1), ..., \omega_j(\gamma_n))'. \tag{7}$$

Definition 3.2. *Given the real-valued stochastic process* $\{x_t, t \geq 1\}$ *and a vector* $\boldsymbol{\delta} \in R^n$, *define the filtered series*

$$\varepsilon_{\boldsymbol{\delta},t} = \Delta_{\boldsymbol{\gamma}}(L; \boldsymbol{\delta})\, x_t, \tag{8}$$

where, if $\boldsymbol{\delta} = \mathbf{d}$, *then* $\Delta_{\boldsymbol{\gamma}}(L; \mathbf{d})\, x_t = \varepsilon_t$ *and* $\varepsilon_{\mathbf{d},t} = \varepsilon_t$. *For any frequency* $\gamma_s \in [0, \pi]$, *define the following (truncated and non-truncated) stochastic processes which are constructed by linearly filtering* $\varepsilon_{\boldsymbol{\delta},t}$ *with the weighting processes given in Definition 3.1:*

$$\varepsilon^{*}_{\gamma_s,t-1} = \sum_{j=1}^{t-1} \omega_j(\gamma_s)\, \varepsilon_{\boldsymbol{\delta},t-j}, \tag{9}$$

$$\varepsilon^{**}_{\gamma_s,t-1} = \sum_{j=1}^{\infty} \omega_j(\gamma_s)\, \varepsilon_{\boldsymbol{\delta},t-j}. \tag{10}$$

Definition 3.3. *Given* $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_n)'$, *define the n-dimensional vectors*

$$\begin{aligned}
\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1} &= \left(\boldsymbol{\varepsilon}^{*}_{\gamma_1,t-1}, ..., \boldsymbol{\varepsilon}^{*}_{\gamma_n,t-1}\right)' = \sum_{j=1}^{t-1} \boldsymbol{\omega}_j(\boldsymbol{\gamma})\, \varepsilon_{\boldsymbol{\delta},t-j}; \\
\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1} &= \left(\boldsymbol{\varepsilon}^{**}_{\gamma_1,t-1}, ..., \boldsymbol{\varepsilon}^{**}_{\gamma_n,t-1}\right)' = \sum_{j=1}^{\infty} \boldsymbol{\omega}_j(\boldsymbol{\gamma})\, \varepsilon_{\boldsymbol{\delta},t-j}.
\end{aligned} \tag{11}$$

3.2 The Lagrange Multiplier test

In this section, we propose a Lagrange Multiplier (LM) type procedure for testing for fractionally integrated patterns. We construct a Gaussian likelihood function, as if the innovations were normally distributed, but noting that our assumptions do not require this condition to ensure the validity of the asymptotic results. The optimizer of this objective function is usually referred to as the quasi-maximum likelihood estimator.

Denote $\boldsymbol{\delta} = \mathbf{d} + \boldsymbol{\theta}$, with i-th element $\delta_i = d_i + \theta_i$. The Gaussian log-likelihood function for $(\boldsymbol{\delta}', \sigma^2)'$, given $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_n)'$ and conditional on the set of information $\mathbf{x}_T = \{x_t, t = -\infty, ..., T\}$ is given by

$$\mathcal{L}(\boldsymbol{\delta}, \sigma^2|\mathbf{x}_T) = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T}(\varepsilon_{\boldsymbol{\delta},t})^2,$$

and, hence, the gradient evaluated under $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$ can be written as

$$\left.\frac{\mathcal{L}(\boldsymbol{\delta}, \sigma^2|\mathbf{x}_T)}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=0} = -\frac{1}{\sigma^2}\sum_{t=1}^{T}\varepsilon_t\left.\left(\frac{\partial\varepsilon_{\boldsymbol{\delta},t}}{\partial\boldsymbol{\theta}}\right)\right|_{\boldsymbol{\theta}=0}.$$

Note for instance that the partial derivative of $\varepsilon_{\boldsymbol{\delta},t}$ on θ_1 is

$$\frac{\partial\varepsilon_{\boldsymbol{\delta},t}}{\partial\theta_1} = \log(1-L)(1-L)^{\theta_1}(1-L)^{d_1}\left[\prod\nolimits_{i=2}^{n-1}\xi_{\gamma_i}(L;\delta_i)\right](1+L)^{\delta_n}x_t$$

which reduces to $\log(1-L)\Delta_{\gamma}(L;\mathbf{d})x_t = \log(1-L)\varepsilon_t$ when the score vector is evaluated at $\boldsymbol{\theta}=\mathbf{0}$. Similarly, the partial derivatives with respect to θ_s, $s=2,...,n-1$, and θ_n, when evaluated under the null hypothesis are given, respectively as,

$$\left.\frac{\partial\varepsilon_{\boldsymbol{\delta},t}}{\partial\theta_s}\right|_{\mathrm{H}_0:\boldsymbol{\theta}=0} = \log\left(1-2\cos\gamma_s L+L^2\right)\varepsilon_t,$$

$$\left.\frac{\partial\varepsilon_{\boldsymbol{\delta},t}}{\partial\theta_n}\right|_{\mathrm{H}_0:\boldsymbol{\theta}=0} = \log(1+L)\varepsilon_t.$$

Following Chung (1996) and Breitung and Hassler (2002), the elements that characterize the score vector under the null hypothesis can be expanded as:

$$\log(1-L)\varepsilon_t = -\sum_{j=1}^{\infty}\left(\frac{1}{j}\right)\varepsilon_{t-j}, \tag{12}$$

$$\log\xi_{\gamma_l}(L;1)\varepsilon_t = -\sum_{j=1}^{\infty}\left(\frac{2\cos(j\gamma_l)}{j}\right)\varepsilon_{t-j}, \tag{13}$$

$$\log(1+L)\varepsilon_t = -\sum_{j=1}^{\infty}\left(\frac{(-1)^j}{j}\right)\varepsilon_{t-j}, \tag{14}$$

which motivates Definition 3.1. Now, by using Definitions 3.2 and 3.3, we can write

$$\left.\frac{\mathcal{L}(\boldsymbol{\delta},\sigma^2|\mathbf{x}_T)}{\partial\boldsymbol{\theta}}\right|_{\mathrm{H}_0:\boldsymbol{\theta}=0} = \frac{1}{\sigma^2}\sum_{t=1}^{T}\varepsilon_t\left(\sum_{j=1}^{\infty}\boldsymbol{\omega}_j\varepsilon_{t-j}\right) \equiv \frac{1}{\sigma^2}\sum_{t=1}^{T}\varepsilon_t\left(\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1}\right) \tag{15}$$

which, under the restriction $\varepsilon_t = 0$, $t\leq 0$ in Assumption $\mathcal{A}$ further reduces to

$$\left.\frac{\mathcal{L}(\boldsymbol{\delta},\sigma^2|\mathbf{x}_T)}{\partial\boldsymbol{\theta}}\right|_{\mathrm{H}_0:\boldsymbol{\theta}=0} = \frac{1}{\sigma^2}\sum_{t=2}^{T}\varepsilon_t\left(\sum_{j=1}^{t-1}\boldsymbol{\omega}_j\varepsilon_{t-j}\right) \equiv \frac{1}{\sigma^2}\sum_{t=2}^{T}\varepsilon_t\left(\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\right). \tag{16}$$

Under the null hypothesis and given the restrictions provided in Assumption $\mathcal{A}$, ε_t is uncorrelated with $\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}$ owing to the MDS property of $\{\varepsilon_t\}$, from which the score has zero expectation. Since $\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1}$ admits a causal representation with square summable coefficients, it therefore follows that $\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}$ is (asymptotically) covariance stationary, and so is the score vector. The Fisher information matrix, estimated as the outer product of gradients, is given by the inverse of

$$\frac{1}{\sigma^4}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_t^2\left(\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},t-1}\right) \tag{17}$$

which converges in probability to a finite, invertible covariance matrix under Assumption $\mathcal{A}$. Therefore, we can devise a suitable test statistic for $\mathrm{H}_0:\boldsymbol{\theta}=\mathbf{0}$ under the Lagrange Multiplier principle. This is formally stated in Theorem 3.1 below.

Theorem 3.1. *Let $\{x_t, t=1,...,T\}$ be an observable process such that Assumption $\mathcal{A}$ holds true. Given some arbitrary $\mathbf{d}\in R^n$, define the test statistic*

$$LM_T = \left(\sum_{t=2}^{T}\varepsilon_{\mathbf{d},t}\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\right)'\left[\sum_{t=2}^{T}\varepsilon^2_{\mathbf{d},t}\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},t-1}\right]^{-1}\left(\sum_{t=2}^{T}\varepsilon_{\mathbf{d},t}\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\right) \tag{18}$$

with $\left\{\varepsilon_{\mathbf{d},t}, \boldsymbol{\varepsilon}^*_{\gamma,t-1}\right\}_{t=1}^T$ *determined on the basis of* $\mathbf{d}$ *according to Definitions 3.1-3.3. Then, under the null hypothesis* $\mathrm{H}_0 : \boldsymbol{\delta} = \mathbf{d}$, or, *equivalently,* $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$*, it follows as* $T \to \infty$ that,

$$LM_T \Rightarrow \chi^2_{(n)}, \tag{19}$$

where $\chi^2_{(n)}$ *stands for a Chi-squared distribution with* n *degrees of freedom.*

Proof. See Appendix.

Theorem 3.1 generalizes the LM test proposed by Tanaka (1999), restricted to the case of a single fractional unit root at the zero frequency ($n = 1, \gamma = 0$), for a single or multiple fractional unit roots at any frequency in $[0, \pi]$, and with innovations which are not necessarily independent but simply MDS. Hence, the testing procedure suggested is robust against (conditional) heteroskedasticity of unknown form provided that the regularity conditions are observed. Under the i.i.d assumption in *ii.a*) the asymptotic variance of the score vector is given by $\sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}$, $\mathbf{\Gamma}_{\boldsymbol{\gamma}} \equiv \sum_{j=1}^{\infty} \boldsymbol{\omega}_j(\boldsymbol{\gamma})\, \boldsymbol{\omega}'_j(\boldsymbol{\gamma})$, which equals $\sigma^2\pi^2/6$ for $\gamma = 0$ and $n = 1$ (see Appendix A for further details on $\mathbf{\Gamma}_{\boldsymbol{\gamma}}$). The variance parameter σ^2 can be estimated consistently as $\widehat{\sigma}^2_T = \sum_{t=2}^{T} \varepsilon^2_{\mathbf{d},t}/T$, where the non-stochastic matrix $\mathbf{\Gamma}_{\boldsymbol{\gamma}}$ can be determined by the close-form representations given in Appendix A for any set of frequencies in $[0, \pi]$, or by simple numerical approximation.

4 Regression-based tests for fractional integration

As an alternative to the previous approach, we can devise testing procedures belonging to the linear regression context which are asymptotically equivalent to the previously discussed LM_T test. The regression based approach was pioneered by Agiakloglou and Newbold (1994) for the context of fractional unit roots at the zero frequency, and further developed in Breitung and Hassler (2002), Hassler and Breitung (2006), and Demetrescu *et al.*, (2007) in the same context. Regression-based tests are particularly advantageous for the empirically relevant case in which the data exhibit week correlation. We discuss the general testing principle and the asymptotic distribution of the relevant tests under the MDS assumption, as well as in the general context of weakly dependent errors.

The following proposition states the general testing strategy for generalized fractional integration in the regression framework:

Proposition 4.1. *Under Assumption* $\mathcal{A}$*, and given* $\{x_t, t = 1, ..., T\}$*, the null hypothesis* $\mathrm{H}_0 : x_t \sim \mathrm{GFI}(\mathbf{d})$, $\mathbf{d} \in R^n$*, can be tested against the alternative* $\mathrm{H}_1 : x_t \sim \mathrm{GFI}(\mathbf{d} + \boldsymbol{\theta})$, $\boldsymbol{\theta} \neq \mathbf{0}$*, through a test for the joint significance of the regression coefficients,* $\{\phi_s\}_{s=1}^n$ *(i.e.,* $\mathrm{H}_0 : \phi_1 = ... = \phi_n = 0$)*, in the following least-squares auxiliary regression:*

$$\varepsilon_{\mathbf{d},t} = \phi_1\varepsilon^*_{\gamma_1,t-1} + \phi_2\varepsilon^*_{\gamma_2,t-1} + ... + \phi_n\varepsilon^*_{\gamma_n,t-1} + e_t. \tag{20}$$

with $\left\{\varepsilon_{\mathbf{d},t}, \varepsilon^*_{\gamma_s,t-1}\right\}_{t=2}^T$ *defined under the null hypothesis as described previously in Definitions 3.1-3.3.*

The OLS estimates $\boldsymbol{\phi}_T = \left(\phi_{1,T}, ..., \phi_{n,T}\right)'$, obtained from the auxiliary regression (20) can be seen as a non-singular transformation of the score vector, which drives the asymptotic distribution of the LM statistic, and which furthermore conveys statistical information about the existing degree of fractional integration in the data. In particular, under the null hypothesis

$$\boldsymbol{\phi}_T = \left(\sigma^2 T^{-1} \sum_{t=2}^{T} \boldsymbol{\varepsilon}^*_{\gamma,t-1} \boldsymbol{\varepsilon}'^*_{\gamma,t-1} \right)^{-1} \left(\frac{1}{T} \frac{\mathcal{L}(\boldsymbol{\delta}, \sigma^2 | \mathbf{x}_T)}{\partial \boldsymbol{\theta}} \bigg|_{\mathrm{H}_0 : \boldsymbol{\theta} = 0} \right) \tag{21}$$

and hence, under Assumption $\mathcal{A}$, $\boldsymbol{\phi}_T \xrightarrow{p} (\sigma^4 \boldsymbol{\Gamma}_\gamma)^{-1} E \left(\frac{1}{T} \frac{\mathcal{L}(\boldsymbol{\delta}, \sigma^2 | \mathbf{x}_T)}{\partial \boldsymbol{\theta}} \Big|_{\mathrm{H}_0 : \boldsymbol{\theta} = 0} \right) = \mathbf{0}$. Therefore, if the null hypothesis, $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$, is true, all the elements in $\boldsymbol{\phi}_T$ are approximately zero in a sufficiently large sample and, hence, testing $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$ with the score test, is asymptotically equivalent to test $\mathrm{H}_0 : \boldsymbol{\phi} = \mathbf{0}$ in this regression framework. The distribution of the relevant tests depends critically on the asymptotic distribution of $\boldsymbol{\phi}_T$. Theorem 4.1 provides the fundamental result in this sense, namely, the asymptotic normality of the estimated coefficients under the set of restrictions considered.

Theorem 4.1. *Let $\boldsymbol{\phi}_T = \left(\phi_{1,T}, ..., \phi_{n,T}\right)'$ be the OLS estimates obtained in the auxiliary regression of Proposition 4.1. Under the null hypothesis $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$, considering Assumption $\mathcal{A}$, and $T \rightarrow \infty$, it follows that*

$$\sqrt{T} \boldsymbol{\phi}_T \Rightarrow \mathcal{N}\left(0, \mathbf{V}_\gamma\right) \tag{22}$$

where

$$\mathbf{V}_\gamma = \left(\frac{1}{\sigma^4}\right) \boldsymbol{\Gamma}_\gamma^{-1} \boldsymbol{\Lambda}_{\varepsilon,\gamma} \boldsymbol{\Gamma}_\gamma^{-1}, \tag{23}$$

with $\boldsymbol{\Gamma}_\gamma = \sum_{j=1}^{\infty} \boldsymbol{\omega}_j(\gamma) \boldsymbol{\omega}'_j(\gamma)$ and $\boldsymbol{\Lambda}_{\varepsilon,\gamma} = E\left(\varepsilon_t^2 \boldsymbol{\varepsilon}^{**}_{\gamma,t-1} \boldsymbol{\varepsilon}'^{**}_{\gamma,t-1}\right)$.

Proof. See Appendix.

Owing to asymptotic normality, and since the null hypothesis only implies linear restrictions on the parameters involved, this can easily be tested by means of a test statistic based on the Wald representation. Note that, although we use the functional form of a Wald-type test, our testing procedure is an LM or score test because it builds directly on the gradient of the likelihood function. Theorem 4.2 discusses its asymptotic distribution.

Theorem 4.2. *Let $\Upsilon_W^{(n)}$ be the Wald-type test statistic defined through the quadratic form*

$$\Upsilon_W^{(n)} = \boldsymbol{\phi}'_T \left[\frac{1}{T} \mathbf{V}_{\gamma,T}\right]^{-1} \boldsymbol{\phi}_T, \tag{24}$$

where $\mathbf{V}_{\gamma,T}$ is the sample analog of $\mathbf{V}_\gamma$ such that

$$\frac{1}{T}\mathbf{V}_{\gamma,T} = \left(\sum_{t=2}^{T} \boldsymbol{\varepsilon}^*_{\gamma,t-1} \boldsymbol{\varepsilon}'^*_{\gamma,t-1}\right)^{-1} \left(\sum_{t=2}^{T} \widehat{e}_t^2 \boldsymbol{\varepsilon}^*_{\gamma,t-1} \boldsymbol{\varepsilon}'^*_{\gamma,t-1}\right) \left(\sum_{t=2}^{T} \boldsymbol{\varepsilon}^*_{\gamma,t-1} \boldsymbol{\varepsilon}'^*_{\gamma,t-1}\right)^{-1} \tag{25}$$

and noting that $\varepsilon^2_{\mathbf{d},t}$ can be used instead of $\widehat{e}_t^2$, where $\widehat{e}_t$ are the empirical residuals. With Assumption $\mathcal{A}$ holding true, under $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$, and as $T \rightarrow \infty$, $\Upsilon_W^{(n)}$ is asymptotically equivalent to LM_T in Theorem 4.1, i.e.,

$$\Upsilon_W^{(n)} \Rightarrow \chi^2_{(n)}.$$

Proof. See Appendix.

Corollary 4.1. *Given* $\mathbf{d} \in R^n$, *inference involving a subset of* m *parameters,* $1 \leq m < n$, *follows similar to Proposition 4.1. Without loss of generality, assume that we are interested on the first* m *long-memory coefficients, thereby assuming that* d_s *for* $s > m$ *is correctly specified. Hence, the alternative hypothesis allows* $\theta_s \neq 0$ *for all* $s \leq m$, *and sets* $\theta_s = 0$, *otherwise. The corresponding auxiliary regression is now given by*

$$\varepsilon_{\mathbf{d},t} = \sum_{s=1}^{m} \phi_s \ \varepsilon^*_{\gamma_s,t-1} + e_{t,m},$$

and the test $\mathrm{H}_0 : \phi_1 = ... = \phi_m = 0$ *is performed in the same terms as in Theorem 4.1, with the Wald-type test statistic now being asymptotically distributed as* $\chi^2_{(m)}$.

Corollary 4.2. *Consider the restricted joint hypothesis* $\boldsymbol{\theta} = \theta \mathbf{1}_n$, with $\theta \neq 0$ *and where* $\mathbf{1}_n$ *is a vector of ones in* R^n. *This is the case, for instance, when analyzing the suitability of so-called (seasonal) rigid models, which assume homogeneity in the order of fractional integration across the set of frequencies involved; see Porter-Hudak (1990) and Hassler (1994). The auxiliary regression is now given by the univariate regression*

$$\varepsilon_{\mathbf{d},t} = \bar{\phi} \ \left(\sum_{s=1}^{n} \varepsilon^*_{\gamma_s,t-1} \right) + u_t,$$

and the relevant statistic, say $\bar{\Upsilon}^{(n)}$, *analyzes the significance of the* $\bar{\phi}$ *parameter. This statistic, which is a squared t-statistic is asymptotically distributed as* $\chi^2_{(1)}$, *since only one restriction is implied.*

Remark 4.1. These LM type tests are asymptotically equivalent to the frequency domain LM tests studied in Robinson (1994), and the time domain LM test considered in Tanaka (1999). The tests are also asymptotically equivalent to the general likelihood-based tests in Nielsen (2004), discussed in the context of maximum-likelihood model estimation. The LM regression-based test in Breitung and Hassler (2002), focusing on the (restricted) fractional unit root model, $\bar{\Delta}_{\boldsymbol{\gamma}} (L; \mathbf{d}) = (1 - L)^d$, arises as a particular case in our context; see also Nielsen (2005), Hassler and Breitung (2006), and Demetrescu *et al.*, (2007). It is worth mentioning that, as remarked in Nielsen (2004), the experimental simulations in Tanaka (1999), and Breitung and Hassler (2002), show that in finite samples the time domain fractional unit-root tests tend to be superior to the frequency domain tests, both in size and power behavior, so a similar performance is likely to be observed in a more general setting as well.

Remark 4.2. The test is robust against conditional heteroskedasticity of unknown form under Assumption $\mathcal{A}$ *ii.b*). This is achieved by using a consistent estimate of the asymptotic covariance matrix $\mathbf{V}_{\boldsymbol{\gamma}}$ based on a version of the Eicker-White estimator as given in (25). If the data are believed to be generated under *ii.a*), then $\mathbf{V}_{\boldsymbol{\gamma}} = \boldsymbol{\Gamma}^{-1}_{\boldsymbol{\gamma}}$, and this may be used directly.

Remark 4.3. As discussed in Breitung and Hassler (2002), the auxiliary regression centered on the zero-frequency, $\varepsilon_{\mathbf{d},t} = \phi_1 \ \varepsilon^*_{0,t-1} + e_t$, is reminiscent of the Dickey-Fuller regression and the

Wald-test in Dolado, Gonzalo and Mayoral (2002). Meaningful differences arise, nevertheless, since in the DF test the regressor is I(0) under the alternative, whereas $\varepsilon^*_{0,t-1}$ is FI$(d+\theta)$ owing to the different types of weights used in constructing these variables. Similarly, for pure seasonal models, the general auxiliary regression in Proposition 4.1 is reminiscent of the Hylleberg, Engle, Granger and Yoo (1990) test regression, in the sense that the regressors $\varepsilon^*_{\gamma_s,t-1}$ are weighted linear combinations of lags of $\varepsilon_{\mathbf{d},t}$ related to a specific seasonal frequency. Further differences arise in this case, because regressors in the HEGY context are ensured to be asymptotically orthogonal by construction, whereas the LM-based regressors are not. This feature advises against testing partial hypothesis (*i.e.*, involving a subset with m parameters) based on the estimates of the general model (i.e., after estimating a regression with $n > m$ parameters), as the covariance matrix is not (block) diagonal. Corollary 4.1 describes the correct way to proceed for this case.

Remark 4.4. Note that, if the auxiliary regression includes a subset of m, $1 \leq m < n$, parameters, the null hypothesis being tested still refers to $\mathbf{d} \in R^n$. In empirical settings, therefore, we can expect subset-testing tending to overreject if any of the long-memory parameters which are not involved in the auxiliary regression is misspecified (even if the null is correct for the parameters included in the regression), because the overall hypothesis is false. Of course, the extent of the size distortion would depend on the degree of autocorrelation in $\varepsilon_{\mathbf{d},t}$ and, hence, on the regression residuals originated by the misspecification. For moderate degrees of autocorrelation, the empirical size could be controlled by resorting to augmented regression (i.e., including lags of the dependent variable) but, in general, large size departures can be expected under naive specifications. We shall discuss this issue more carefully in the Monte Carlo section.

Remark 4.5. Generalized fractional integrated models are particularly difficult to estimate in practical settings owing to their strong non-linear nature. Proposition 4.1 provides a valuable tool to construct confidence sets that include the true value, say $\mathbf{d}_0 \in R^n$, with $(1-\alpha)\,\%$ asymptotic nominal probability. These sets could be used to obtain reliable starting values for optimization routines aiming to estimate $\mathbf{d}_0$, such as the (quasi)-maximum likelihood methods discussed in Chung (1996) and Nielsen (2004). Confidence sets obtain from a grid-search in Θ, a compact subset of R^n, by using the results in Proposition 4.1. For instance, denote $\Upsilon^{(n)}_{W,\mathbf{d}}$ as the value of the test statistic in Theorem 4.2 when evaluated at any $\mathbf{d} \in \Theta$, and let $\mathcal{D}_{T,\alpha} = \left\{\mathbf{d} : \Pr\left[\chi^2_{(n)} \leq \Upsilon^{(n)}_{W,\mathbf{d}}\right] \leq 1-\alpha\right\}$, *i.e.*, the subset of Θ containing all the vectors for which the null hypothesis cannot be rejected at the $(1-\alpha)\,\%$ asymptotic nominal confidence level. If $\mathcal{D}_{T,\alpha}$ is in the interior of Θ, then the probability of $\mathbf{d}_0$ being in the closure of $\mathcal{D}_{T,\alpha}$ is at least $(1-\alpha)\,\%$. The grid-search process is computational feasible because n is not large in empirical models, and because long-memory parameters usually take values in a small range. For rigid models, a confidence interval of the form $\left[d^\alpha_{T,l}, d^\alpha_{T,u}\right]$ can easily be constructed from Corollary 3.2, given $\bar{\mathcal{D}}_{T,\alpha} = \left\{d : \Pr\left[\chi^2_{(1)} \leq \bar{\Upsilon}^{(n)}_d\right] \leq 1-\alpha\right\}$, by setting $d^\alpha_{T,l} = \inf \bar{\mathcal{D}}_{T,\alpha}$ and $d^\alpha_{T,u} = \sup \bar{\mathcal{D}}_{T,\alpha}$.

Remark 4.6. Throughout our analysis, we have assumed that the vector of frequencies, $\boldsymbol{\gamma}$, is known. Indeed, this is the case for pure seasonal models, but in general terms it may result restrictive when analyzing cyclical models by means of Gegenbauer polynomials. Several approaches have been proposed to estimate Gegenbauer-frequencies consistently in the semi-parametric literature; see, among others, Yajima (1996), Giriatis, Hidalgo, and Robinson (2001), Hidalgo and Soulier (2004), and Hidalgo (2005). In any case, when using sample estimates for subsequent inference purposes, it should be noticed that the performance of the test statistics may be subject to potential distortions that often arise as a result of (small-sample) biases when inferring the unknown elements of $\boldsymbol{\gamma}$.

Example: To illustrate the general testing principle we consider the pure seasonal quarterly case. Assume that the interest lies in testing the suitability of the seasonal unit root model, $(1-L^4)\,x_t = \varepsilon_t$, against a more general case in which the order of seasonal integration is possibly a non-integer value $1+\theta$, $\theta \neq 0$, but believed to be common for all frequencies, *i.e.*, $(1-L^4)^{1+\theta}\,x_t = \varepsilon_t$. Therefore, we have $\boldsymbol{\gamma} = (0, \pi/2, \pi)'$, $n = 3$, and the testing procedure for the rigid seasonal model is that described in Corollary 4.2. Thus, we first compute $\{\varepsilon_{\mathbf{d},t}\}$ by differencing the series under the null hypothesis, *i.e.*, $\varepsilon_{\mathbf{d},t} = x_t - x_{t-4}$, and then compute the regressor $\bar{\varepsilon}_{\gamma,t} = \varepsilon^*_{0,t-1} + \varepsilon^*_{\pi/2,t-1} + \varepsilon^*_{\pi,t-1}$, as discussed previously. Note that

$$\begin{aligned}\bar{\varepsilon}_{\gamma,t} &= \sum_{j=1}^{t-1}\left(\frac{1}{j} + \frac{(-1)^j}{j} + \frac{2\cos(j\pi/2)}{j}\right)\varepsilon_{\mathbf{d},t-j} \\ &= \sum_{j=1}^{t-1}\frac{\varepsilon_{\mathbf{d},t-4j}}{j}\end{aligned}$$

with $\bar{\varepsilon}_{\gamma,t} = 0$ for all $t \leq 0$, so the weighting scheme applied to construct $\bar{\varepsilon}_{\gamma,t}$, namely, $(j^{-1}L^4)$, corresponds to the expansion of $\log[(1-L^4)]$, which by construction ensures power against quarterly seasonal fractional integration. If the data are normally-distributed, this test is fully efficient. Furthermore, a confidence interval of the form $\left[d^{\alpha}_{T,l}, d^{\alpha}_{T,u}\right]$ for the true value of the long-memory parameter under the assumption of homogeneous integration can readily be constructed.

4.1 Short-memory dynamics in mean

Assumption $\mathcal{A}$ imposes uncorrelated errors in the DGP, which may be a restrictive assumption for many empirical applications. In order to generalize the approach to allow for weakly correlated errors, we introduce the following generalization of Assumption $\mathcal{A}$:

Assumption $\mathcal{B}$:
(a) The observable process is generated as $\Delta_{\boldsymbol{\gamma}}(L;\mathbf{d})\,x_t = \varepsilon_t \mathbb{I}_{(t>0)}$, *satisfying the conditions in Assumption* $\mathcal{A}$*i);*
(b) The innovation process satisfies $a(L)\,\varepsilon_t = v_t$, *where* $a(L) = 1 - \sum_j^p a_j L^j$, $p \geq 0$, *such that* $a(z)$ *has all its roots outside the unit circle.*
(c) The innovation process $\{v_t, \mathcal{F}_t\}$, $\mathcal{F}_t = \sigma(v_j : j \leq t)$, *is a stationary and ergodic MDS,* $E(v_t^2) = \sigma^2$, and $\{v_t\}$ *satisfies the restrictions in either Assumption* $\mathcal{A}$ *iia)* or *iib).*

Assumption $\mathcal{B}$ allows for stationary AR(p) dynamics in the generating process, which may appear jointly with time-varying volatility patterns, such as GARCH or Stochastic Volatility errors, under the same set of restrictions as those in Assumption $\mathcal{A}$. The remaining proofs are formally discussed for the case in which p is known. For practical purposes, the short-run dynamics of the underlying process may be characterized by a stationary and invertible linear process $\varepsilon_t = \sum_{j=0}^{\infty} b_j v_{t-j}$ such that the AR(p) model, for some large enough $p < \infty$, approaches the underlying AR representation reasonably well. The effects on the finite-sample properties of the regression-based tests when the underlying correlation structure in the short-run dynamics is unknown shall be discussed in the Monte Carlo section.

Proposition 4.2. *Consider the basic auxiliary regression in Proposition 4.1 augmented with p lags of the dependent variable, i.e.,*

$$\varepsilon_{\mathbf{d},t} = \sum_{l=1}^{n} \phi_l \varepsilon^*_{\gamma_l,t-1} + \left(\sum_{i=1}^{p} \zeta_i \varepsilon_{\mathbf{d},t-i} \right) + e_{tp}, \quad t = p+1, ..., T. \tag{26}$$

Then, the null hypothesis $H_0 : \boldsymbol{\theta} = \mathbf{0}$ can be tested by addressing the joint significance of the estimated ϕ_l coefficients in the augmented auxiliary regression.

Augmentation is standard in many testing procedures having the null of (fractional) integration. Among these, the most well-known case is the Augmented Dickey-Fuller unit root test; see also Dolado, Gonzalo and Mayoral (2002) and Breitung and Hassler (2002) for augmentation under the null of fractional integration. Essentially, augmenting the auxiliary regression with lags of the dependent variable seeks to whiten the correlation structure of the regression residuals so that they can behave asymptotically as a MDS. From this, the relevant test statistic is expected to retain asymptotic invariance, and the same critical values discussed under uncorrelated errors hold in this context as well. The following theorems present the asymptotic properties of the regression based test statistic for general fractional integration.

Theorem 4.3. *Let $\boldsymbol{\beta}_T$ be the (n+p) estimated vector of parameters in the pth order augmented auxiliary regression $\varepsilon_{\mathbf{d},t} = \boldsymbol{\beta}'\mathbf{X}^*_{tp} + e_{tp}$, with $\mathbf{X}^*_{tp} = \left(\boldsymbol{\varepsilon}'^*_{\boldsymbol{\gamma},t-1}, \varepsilon_{\mathbf{d},t-1}, ..., \varepsilon_{\mathbf{d},t-p}\right)'$, and let the (n+p) vector $\boldsymbol{\mu}_0 = (0, ..., 0, a_1, ..., a_p)'$, with the a_i parameters corresponding to the autoregressive coefficients in $(1 - \sum_{i=1}^{p} a_i L)\, \varepsilon_t = v_t$. Then, under Assumption $\mathcal{B}$, the null hypothesis, and as $T \to \infty$,*

$$\sqrt{T}\,(\boldsymbol{\beta}_T - \boldsymbol{\mu}_0) \Rightarrow \mathcal{N}\left(\mathbf{0},\ \left(\boldsymbol{\Omega}^{**}_p\right)^{-1} \boldsymbol{\Lambda}_p \left(\boldsymbol{\Omega}^{**}_p\right)^{-1}\right) \tag{27}$$

with $\boldsymbol{\Omega}^{**}_p \equiv E\left(\mathbf{X}^{**}_{tp}\mathbf{X}'^{**}_{tp}\right)$ and $\boldsymbol{\Lambda}_p \equiv E\left(v_t^2 \mathbf{X}^{**}_{tp}\mathbf{X}'^{**}_{tp}\right)$, where $\mathbf{X}^{**}_{tp} = \left(\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1}, \varepsilon_{\mathbf{d},t-1}, ..., \varepsilon_{\mathbf{d},t-p}\right)'$.

Proof. *See Appendix.*

Theorem 4.4. *Let $\mathbf{R}$ be an $n \times (n+p)$ matrix such* that $[\mathbf{R}]_{ij} = 1$ *for all $i = j$ and zero otherwise. Consider the Wald-type test statistic on the estimates of the augmented auxiliary regression, i.e.,*

$$\Upsilon^{(n)}_{Wp} = [\mathbf{R}\boldsymbol{\beta}_T]' \left[\frac{1}{T}\mathbf{R}\widehat{\mathbf{V}}_{\mathbf{T}}\mathbf{R}'\right]^{-1} [\mathbf{R}\boldsymbol{\beta}_T] \tag{28}$$

with $\widehat{\mathbf{V}}_{\mathbf{T}}$ being the sample estimation of the covariance matrix of $\boldsymbol{\beta}_T$ such that

$$\widehat{\mathbf{V}}_{\mathbf{T}}\ /T = \left(\sum_{t=p+1}^{T} \mathbf{X}^*_{tp}\mathbf{X}^{*\prime}_{tp}\right)^{-1} \left(\sum_{t=p+1}^{T} \widehat{e}^{\,2}_{tp}\mathbf{X}^*_{tp}\mathbf{X}^{*\prime}_{tp}\right) \left(\sum_{t=p+1}^{T} \mathbf{X}^*_{tp}\mathbf{X}^{*\prime}_{tp}\right)^{-1},$$

where $\widehat{e}_{tp}$ denotes the estimated residuals. Under the same conditions of Theorem 4.3, $\Upsilon_{Wp}^{(n)}$ is asymptotically equivalent to LM_T, i.e., $\Upsilon_{Wp}^{(n)} \Rightarrow \chi^2_{(n)}$.

Proof. *See Appendix.*

Corollary 4.3. *If $\{v_t\}$ is i.i.d with finite fourth-order moment, $E\left(v_t^2 \mathbf{X}_{tp}^{**}\mathbf{X}_{tp}^{**\prime}\right) \propto E\left(\mathbf{X}_{tp}^{**}\mathbf{X}_{tp}^{**\prime}\right)$. Hence, the null hypothesis H_0: $\phi_1 = ... = \phi_n = 0$ can easily be tested by using alternative test statistics which can be constructed under the Lagrange Multiplier and Likelihood Ratio principles, and which are asymptotically-equivalent to LM_T. As discussed previously, in the context of this paper all these tests are necessarily LM tests regardless of their functional form. Let $\Upsilon_{LR,p}^{(n)} = T\left(\log \mathcal{S}_R - \log \mathcal{S}_u\right)$ and $\Upsilon_{LM,p}^{(n)} = T\left(\mathcal{S}_R - \mathcal{S}_u\right)/\mathcal{S}_R$, where $\mathcal{S}_R$ and $\mathcal{S}_u$ denote the squared sum of restricted and unrestricted residuals, respectively. Then, under the null, and as $T \rightarrow \infty$, $\Upsilon_{LR,p}^{(n)} \Rightarrow \chi^2_{(n)}$* and *$\Upsilon_{LM,p}^{(n)} \Rightarrow \chi^2_{(n)}$.*

Corollary 4.4. *The same considerations as in Corollaries 4.1 and 4.2 apply when using an augmented test regression.*

Remark 4.7 Demetrescu *et al.* (2007) analyze the performance of several procedures to determine the order of augmentation, p, in finite samples. Whereas data-dependent selection procedures exhibit a poor performance, it is found that the rule of thumb proposed by Schwert (1989) shows relatively good performance in finite-samples. This sets $p = \left[c(T/100)^{1/4}\right]$, where c is a positive constant and $[\cdot]$ denotes the integer value of the argument.

Remark 4.8. We have focused on the model $\Delta_\gamma\left(L; \mathbf{d}\right)\left(x_t - \mu_t\right) = \varepsilon_t \mathbb{I}_{(t>0)}$, by allowing different dynamics in ε_t, and restricting $\mu_t = 0$. As commented in Breitung and Hassler (2002), the simplest way to deal with non-zero deterministic patterns, $\mu_t \neq 0$, is to detrend x_t prior to computing the relevant tests statistics. This does not affect the limit distribution of the relevant statistics; see the discussion in Robinson (1994).

Remark 4.9. The theoretical derivation of the local power functions under the alternative is a nontrivial problem due to the multiple hypothesis context. For restricted cases, it becomes more tractable, and it can be shown, following for instance Tanaka (1999) and Demetrescu *et al.* (2007) that the test procedures will converge to a noncentral chi-squared distribution under local alternatives. Since for applied purposes the behavior of the power function in finite-samples is particularly relevant, we shall address this issue carefully in the Monte Carlo section.

5 Finite-sample analysis

In this section we address the empirical properties of the regression-based LM test statistic in finite samples. The case for the zero-frequency fractionally-integrated unit root process, $\bar{\Delta}_\gamma\left(L; \boldsymbol{\delta}\right) = \left(1 - L\right)^{d_0}$, has received considerable attention in literature; see for instance, Breitung and Hassler (2002), and Nielsen (2004), among others. These show the good finite-sample performance of LM tests, both in absolute terms and in relation to alternative frequency domain based procedures. We therefore analyze cyclical and seasonal models aiming to contribute to better understand the properties of LM tests in the general context.

The applied literature on cyclical or seasonal fractionally-integrated models has focused on both economic and non-economic variables. Empirical datasets are characterized by quite different features. The number of observations available for financial and many geophysical variables is relatively large, and often includes several thousands observations, whereas the length of macroeconomic variables is much more limited.[1] Data recorded on a high-frequency basis typically exhibit persistent short-run dynamics, whereas aggregated data tend to display considerably weaker forms of serial dependence. We consider the possibility of different types of short-run dynamics as well as different sample sizes to analyze the empirical size and power. In particular, we focus on samples of length $T = \{100, 250, 500\}$. For datasets involving a large number of observations, as some of those analyzed in applied literature, the asymptotic theory is expected to provide a good approximation.

In the first experiment we consider a simple pure cyclical model,

$$(1 - 2\cos\gamma_s L + L^2)^{d+\theta} x_t = \varepsilon_t$$

in order to analyze the empirical size and power of $\Upsilon_W^{(1)}$, asymptotically distributed as $\chi^2_{(1)}$, when testing $\mathrm{H}_0 : d = 1$ with true values given by $d = 1$ and θ in $[-0.3, 0.3]$. We consider 5000 replications and $\varepsilon_t \sim iid\mathcal{N}(0, 1)$. Since the Gegenbauer frequency γ_s is a 'free' parameter, we set $\gamma_s = s\pi/10$, with $s = 1, ..., 9$. The rejection frequencies for a nominal significance level of 5% and sample sizes of $T = 100$ and $T = 250$ are shown in Table 1.

The test shows approximately correct size and good power performance even in small samples. Only minor differences, following no particular pattern, arise across the frequencies λ_s considered. For non-zero vales of θ, we observe several interesting features in the empirical power functions. First, given λ_s and T, power tends to exhibit a symmetric U-shape figure around the $\pi/2$ frequency, which is more evident for small values of $|\theta|$. This suggest that, the larger the difference $|\gamma_s - \pi/2|$ with $\gamma_s \in (0, \pi)$, the more powerful the testing procedure becomes. The dependence of power on the particular frequency the test is related to is not surprising, since the variance of the regressor (and hence, the signal-to-noise ratio and, eventually, the power of the test) depends on the specific frequency, γ, considered and, more generally, on $\boldsymbol{\gamma}$; see Appendix A for further technical details. Furthermore, if we compare these results to those in Breitung and Hassler (2002, Table 1, p.176) for the zero-frequency case, the power observed at the long-run frequency is approximately of the same order as that for $\gamma = \pi/2$. This suggest that, everything else equal, fractionally-integrated dynamics are generally easier detected at the cyclical than at the zero-frequency. A similar feature appears when dealing with $\gamma = \pi$ (not reported here) for which power is similar to that of $\gamma = \pi/2$.[2] Dealing with the non-zero frequency also has other benefits in terms of power. For fixed T and γ_s, the power functions tend to be symmetric around $\theta = 0$, since only the size of $\theta - 0$, and not its sign,

[1]The dataset in Bouette *et al.* (2006), refering to hourly average wind speeds measured between 1951 and 2003, includes over 16,000 observations. Soares and Souza (2006) consider two years of hourly electricity demand. Gil-Alana (2005) studies US monthly inflation in a dateset with more than 1000 observations.

[2]Note that the asymptotic variance is proportional to $\psi(\gamma)$, see Appendix A. This is a positive, symmetric and non-continuous function on $[0, \pi]$ that takes minimum value $\psi(\gamma) = \pi^2/6$ for $\gamma = \{0, \pi/2, \pi\}$, and maximum value given by $\lim_{\lambda\to 0^-}\psi(\gamma)$ $=\lim_{\lambda\to\pi^+}\psi(\gamma) = 2\pi^2/3$. We therefore can expect a discontinuity in the power function for the case $\gamma = 0 + \epsilon$ or $\lambda = \pi -\ \epsilon$ even for an arbitrarily small $\epsilon > 0$.

seems to drive the probability of rejection. This does not seem to be the case for the zero-frequency case analyzed in Breitung and Hassler (2002), where the LM test is likely to reject more easily if $\theta < 0$. Finally, power is largely enhanced even for a small sample of $T = 250$, and virtually reaches 100% for all the tests when $T = 500$, thus showing the consistency of the testing procedure even in small samples.

Second, we consider a more general two-factor cyclic model given by,

$$(1 - 2\cos\gamma_1 L + L^2)^{d_1+\theta_1}(1 - 2\cos\gamma_2 L + L^2)^{d_2+\theta_2} x_t = \varepsilon_t.$$

We want to address the ability of the unrestricted joint test $\Upsilon_W^{(2)}$, distributed asymptotically as $\chi^2_{(2)}$, as well as the joint restricted test $\bar{\Upsilon}^{(2)}$ discussed in Corollary 4.2, and individual squared t-tests, say $\Upsilon_{\gamma_1}^{(1)}$ and $\Upsilon_{\gamma_2}^{(1)}$, distributed asymptotically as $\chi^2_{(1)}$, to detect fractionally-integrated dynamics. Subset-testing is discussed in Corollary 4.1. As before, we set $d_1 = d_2 = 1$, and θ_1, θ_2 in $[-0.3, 0.3]$, considering 5000 replications and $\varepsilon_t \sim iid\mathcal{N}(0,1)$. The joint test $\Upsilon_W^{(2)}$ is expected to reject the null hypothesis if fractional integration is present in, at least, one of the frequencies involved, while the individual tests may only reject when fractional integration occurs at the frequency they are related to. The restricted joint test $\bar{\Upsilon}^{(2)}$ should be more efficient than $\Upsilon_W^{(2)}$ when the restriction $\theta_{\gamma_1} = \theta_{\gamma_2}$ is true, but it is expected to exhibit less comparative power to reject the false null otherwise.

In view of the previous experiment, we expect the power function to depend on the value of $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)'$. We set $\gamma_1 = 0.15 \approx \pi/20$, corresponding to the estimated frequency of the business cycle by the NBER, and consider what seems to be the most unfavorable frequency for the tests when dealing with frequencies in $(0, \pi)$, given by $\gamma_2 = \pi/2$, which also corresponds to one of the harmonics of quarterly and monthly seasonality. For frequencies $\gamma \in (0, \pi)$ away from $\pi/2$, further simulations (not reported here) showed a much better statistical performance both in terms of size and power. The rejection frequencies for a nominal significance level of 5% and sample length $T = 100$ are shown in Table 2.

Several interesting features emerge from this experiment. First, we comment the results for the individual tests $\Upsilon_{0.15}^{(1)}$ and $\Upsilon_{\pi/2}^{(1)}$. When $d_1 = 1$, and $d_2 = 1 + \theta_2$, both tests have approximately correct size when θ_2 is close to zero. However, when $|\theta_2|$ moves away from the origin, $\Upsilon_{0.15}^{(1)}$ may show size departures with respect to the nominal size, which are particularly important when $\theta_2 > 0$. This is also true for the $\Upsilon_{\pi/2}^{(1)}$ test when $d_2 = 1$ and $d_1 = 1 + \theta_1$, now noting massive size distortions for large $\theta_1 > 0$. As remarked in Section 4, these distortions are originated from residual autocorrelation resulting from misspecification. For moderate degrees of autocorrelation, size departures can be considerably reduced by augmenting the auxiliary regression with $\bar{p}$ lags of the dependent variable. Table 1 shows, for $\bar{p} = 2$, that augmentation is effective in reducing the distortion, particularly in the region $\theta > 0$ in which the effect was more pronounced. However, as usual, empirical size is corrected at the cost of power reductions, which in this context can be large for the alternatives $\theta > 0$. Finally, it is interesting to note that, when the empirical size approaches the asymptotic nominal level (correct specification), the power of the $\Upsilon_{\pi/2}^{(1)}$ test is only slightly smaller than that observed when the DGP only includes a Gegenbauer polynomial. Similar behavior can be observed for $\Upsilon_{0.15}^{(1)}$.

In relation to the joint test statistics $\bar{\Upsilon}^{(2)}$ and $\Upsilon_W^{(2)}$, we observe that the restricted test is more powerful than the latter when the restriction $\theta_1 = \theta_2$ is true, but it is also considerably less efficient in the general context $\theta_1 \neq \theta_2$, particularly for small values of $|\theta|$. Both tests tend to reject more easily the (false) null when fractional integration is present at the frequency 0.15, *i.e.*, at the frequency for which the magnitude $|\lambda_s - \pi/2\ |$ is larger. For instance, if $d_1 = 1 - 0.1$ and $d_2 = 1$, the power of $\bar{\Upsilon}^{(2)}$ and $\Upsilon_W^{(2)}$ is, approximately, 39.8% and 48.7%, respectively. In contrast, for $d_1 = 1$ and $d_2 = 1 - 0.1$, the power is only 8.2% and 16.1%. When both θ_1 and θ_2 move away from the origin, the power of joint tests, particularly that of $\Upsilon_W^{(2)}$, largely increases. We note that the power of $\Upsilon_W^{(2)}$ seems to be symmetric for the set of frequencies considered, whereas $\bar{\Upsilon}^{(2)}$ tends to reject more easily when $\theta_1 > 0$ and $\theta_2 < 0$ than in the converse case. For instance, the power of $\Upsilon^{(2)}$ for $\theta_1 = 0.3$ and $\theta_2 = -0.3$ is almost 100%, whereas it is around 25% for $\theta_1 = -0.3$ and $\theta_2 = 0.3$. By sharp contrast, the power of the unrestricted test $\Upsilon_W^{(2)}$ in any of these cases is almost 100%. Finally, and as in the case of the one-factor model, considering larger samples, $T = \{250, 500\}$, leads to considerable improvement of the statistical properties of all the tests. We do not present these results to save space, but these are available upon request.

Finally, the last set of experiments considers again the two-factor filter $\Delta_{\gamma}(L; \boldsymbol{\delta}) = (1 - 2\cos\gamma_1 L + L^2)^{d_1+\theta_1}(1 - 2\cos\gamma_2 L + L^2)^{d_2+\theta_2}$ now allowing for stationary and invertible ARMA patterns in the error term, *i.e.*, we analyze the performance of the augmented-based test statistics when the DGP is,

$$\begin{aligned} \Delta_{\gamma}(L; \boldsymbol{\delta})\, x_t &= \varepsilon_t \\ (1 - aL)\, \varepsilon_t &= (1 - bL)\, v_t, \end{aligned}$$

under the restriction $|a| < 1$ and $|b| < 1$. We first focus on ARMA(1,1) dynamics and, as in Demetrescu *et al.* (2007), set $a = 0.5$ and $b = -0.5$. The ARMA(1,1) model is particularly relevant because short-run dynamics in empirical applications are usually characterized parsimoniously through this specification. Additionally, we analyze in more detail the effects of persistence through an AR(1) with parameter $a = \{0.5, 0.75, 0.9\}$ and $b = 0$ in the above specification. Since for empirical purposes the underlying structure of the short-run component is typically unknown, we explore the effects on the tests when the number of lags to be included in the auxiliary regression are determined according to Schwert's rule, $p = \left[4(T/100)^{1/4}\right]$, as this showed the best performance in the empirical analysis in Demetrescu *et al.* (2007). The rejection frequencies for the individual and joint tests given ARMA(1,1) patterns for $T = \{100, 500\}$ are shown in Table 3, whereas Tables 4 and 5 report the respective empirical results for the AR(1) errors given the values of the autoregressive coefficient a.

We first comment the results for the ARMA(1,1) dynamics. The general conclusions that arise for the weakly-dependent case are similar to those observed for the i.i.d case, although we observe several quantitative changes. Augmentation proves able to help correct the empirical size for all tests, and only small undersizing effects are observed in our simulations. However, and as shown in previous literature, ensuring correct empirical size against general ARMA dynamics through augmentation in small samples, such as $T = 100$, comes usually at the cost of potentially large reductions in power in relation to the i.i.d. case. This pervasive effect has been widely documented in the unit-root literature, where the augmented Dickey-Fuller regression is probably the most widely used in applied settings. In fact, the power of the individual and the joint tests shows figures similar in magnitude to those observed in Demetrescu *et al.* (2007) for the fractional unit root case. By sharp contrast to the unit-root case, fortunately, power improves considerably faster at frequencies away from zero. For instance, for the ARMA model considered, the power of $\Upsilon^{(2)}_{W,p}$ is not larger than 39% in the range of $\boldsymbol{\theta}$ considered when only 100 observations are available, corresponding to $\boldsymbol{\theta}^* = (-0, 3, 0.3)'$. For a larger sample of $T = 500$, everything else equal, power increases up to 98%. Similarly, the joint restricted test $\bar{\Upsilon}^{(2)}_p$ has a peak of approximately 30% for $T = 100$ when $\theta_1 = \theta_2 = -0.3$, which dramatically increases up to 99% for a sample of 500 observations. Finally, $\Upsilon^{(1)}_{0.15}$ and $\Upsilon^{(1)}_{\pi/2}$ have power of 43% and 28% under $\boldsymbol{\theta}^*$ when $T = 100$, respectively, whereas for $T = 500$ power reaches 95% and 83%, respectively.

Similar results can be seen when analyzing the effects of persistence in residuals. Although the empirical size is approximately correct in all cases, as the autoregressive root approaches one in a small sample with 100 observations, power reductions with respect to the i.i.d. case are far more evident. For small values of $|\boldsymbol{\theta}|$ it becomes difficult to reject the false null, and even for some configurations which include relatively large values of $\boldsymbol{\theta}$ when $a = 0.9$. As in the previous case, the power of the tests considerably improves as the number of observations increases. Therefore, for the test $\Upsilon^{(2)}_{W,p}$, given the set of observations that is typically available for many empirical applications, augmenting the regression proves a valid tool to ensure empirical sizes close to the asymptotic nominal level and good power properties.

6 Conclusion

In this paper, we have considered a regression-based LM test in the time-domain that allows testing for fractionally-integrated patterns against integer integration in general models. The tests involving single or multiple parameters can be computed from simple least-squares regressions, and are asymptotically equivalent to the frequency-domain LM test of Robinson (1994) and the likelihood-based tests in Nielsen (2004), for which the relevant critical values obtain from a χ^2 distribution with as many degrees of freedom as the restrictions being tested, and independent of the order of integration. Augmented versions of these tests are asymptotically robust against weakly-dependent errors following unknown patterns under quite general conditions, and exhibit good statistical performance in samples of moderate size. This makes the general regression-based LM testing strategy discussed in this paper a valuable tool for addressing preliminary data analysis in which parsimonious yet potentially restrictive hypothesis related to the order of integration of the data may be formally validated or refuted.

Appendix A: Limit Processes

Consider the limit expressions which characterize the asymptotic variances and covariances of the partial sum processes under i.i.d observations.

Definition A. *For any* $\gamma \in [0, \pi]$ *, let*

$$\psi(\gamma) = \lim_{T\to\infty} \sum_{j=1}^{T} \omega_j^2(\gamma).$$

Straightforward calculus shows that $\psi(\gamma) = \pi^2/6$, if $\gamma \in \{0, \pi\}$, and $\psi(\gamma) = 2(\pi^2/3 - \pi\gamma + \gamma^2)$ *, otherwise. Similarly, given* $\gamma_n, \gamma_m \in [0, \pi]$*,* $\gamma_n \neq \gamma_m$*, let*

$$\psi(\gamma_n, \gamma_m) = \lim_{T\to\infty} \sum_{j=1}^{T} \omega_j(\gamma_n)\, \omega_j(\gamma_m).$$

Note that $|\psi(\gamma_n, \gamma_m)| < \infty$ *and, in particular,*

$$\psi(\gamma_n, \gamma_m) = \begin{cases} -\psi(\gamma_m)/2 & if\ \gamma_n = 0, \gamma_m = \pi \\ (\psi(\gamma_m) - \gamma_m^2)/2 & if\ \gamma_n = 0, \gamma_m \in (0, \pi) \\ (\gamma_m^2 - \psi(\gamma_m))/4 & if\ \gamma_n = \pi, \gamma_m \in (0, \pi) \end{cases},$$

and, if $\gamma_n, \gamma_m \in (0, \pi)$ *, then*

$$\begin{aligned} \psi(\gamma_n, \gamma_m) &= 2\pi/3 - \pi(\gamma_n + \gamma_m + |\gamma_n - \gamma_m|) \\ &\quad + \left((\gamma_n + \gamma_m)^2 + (|\gamma_n - \gamma_m|)^2\right)/2. \end{aligned}$$

Definition B. *Given* $\boldsymbol{\gamma} \equiv (\gamma_1, ..., \gamma_n)'$, with $0 = \gamma_1 < \gamma_2 < ... < \gamma_n = \pi$*, denote* $\boldsymbol{\Gamma}_{\boldsymbol{\gamma}} = \lim_{T\to\infty} \sum_{j=1}^{T} \boldsymbol{\omega}_j(\boldsymbol{\gamma})\, \boldsymbol{\omega}_j(\boldsymbol{\gamma})'$, i.e.,

$$\boldsymbol{\Gamma}_{\boldsymbol{\gamma}} = \begin{pmatrix} \psi(\gamma_1) & \psi(\gamma_1, \gamma_2) & ... & \psi(\gamma_1, \gamma_n) \\ \psi(\gamma_2, \gamma_1) & \psi(\gamma_2) & ... & \psi(\lambda_2, \gamma_n) \\ \vdots & \vdots & \ddots & \vdots \\ \psi(\gamma_n, \gamma_1) & \psi(\gamma_n, \gamma_2) & ... & \psi(\gamma_n) \end{pmatrix},$$

with $\boldsymbol{\Gamma}_{\boldsymbol{\gamma}} < \infty$ *being a symmetric positive definite matrix.*

Appendix B: Technical Proofs

Before proceeding, consider the following additional notation. For an $(n \times 1)$ vector A, $||A||$ denotes the Euclidean vector norm, such that $||A||^2 = A'A$. For an $(n \times m)$ matrix A, $||A||$ denotes the Euclidean matrix norm, $||A||^2 = tr\left(A'A\right)$. The constant K is used throughout the proofs to refer to some generic strictly positive constant which does not depend on the sample size. The notation, $\Rightarrow$, $\overset{p}{\rightarrow}$, $\overset{ms}{\rightarrow}$, $\rightarrow$ denotes weak convergence, convergence in probability, mean square convergence and convergence of a series of real numbers, respectively. The conventional notation $o(1)$ $(o_p(1))$ is used to represent a series of numbers (random numbers) converging to zero (in probability), while $O(1)$ $(O_p(1))$ denotes a series of numbers (random numbers) that are bounded (in probability). As in the main text, $\mathbb{I}_{(\cdot)}$ is an indicator function, and vectors and matrices are denoted through bold letters. Finally, since $\boldsymbol{\gamma}$ is used to refer to the vector of frequencies that characterize the filter $\Delta_{\boldsymbol{\gamma}}(L;\boldsymbol{\delta})$, we shall use the short-hand notation $\boldsymbol{\omega}_j \equiv \boldsymbol{\omega}_j(\boldsymbol{\gamma})$ as there is no risk of confusion.

Proofs for uncorrelated errors

Lemma B.1. *Let* $\varepsilon_t = \Delta_{\boldsymbol{\gamma}}(L;\mathbf{d})\, x_t$ *and* $\boldsymbol{\gamma} \equiv (\gamma_1, .., \gamma_n)'$. *Consider the random vectors,* $\boldsymbol{\varepsilon}^*_{\boldsymbol{\gamma},t-1}$ *and* $\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1}$ *as given in Definition 3.3, and let* $\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1} - \boldsymbol{\varepsilon}^*_{\boldsymbol{\gamma},t-1} = \boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}$, and $\boldsymbol{\Omega}^{**}_t = \boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1}$, $\boldsymbol{\Omega}^*_t = \boldsymbol{\varepsilon}^*_{\gamma,t-1}\boldsymbol{\varepsilon}'^*_{\gamma,t-1}$. *Then, for any arbitrary constants* $\alpha > 0$, $\beta > 1/2$, *it follows as* $T \rightarrow \infty$ that,

i) $\boldsymbol{\vartheta}_{\boldsymbol{\gamma},t-1} = O_p\left(t^{-1/2}\right)$, and $E||\varepsilon_t \boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}||^2 = O\left(t^{-1}\right) + o\left(t^{-2}\right)$,
ii) $||T^{-\alpha}\sum_{t=2}^{T}\varepsilon_t\boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}|| = o_p(1)$ and $||T^{-\alpha}\sum_{t=2}^{T}\boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}|| = o_p(1)$,
iii) $||T^{-\beta}\sum_{t=2}^{T}\left(\boldsymbol{\Omega}^{**}_t - \boldsymbol{\Omega}^*_t\right)|| = o_p(1)$,
iv) $||T^{-\beta}\sum_{t=2}^{T}\varepsilon_t^2\left(\boldsymbol{\Omega}^{**}_t - \boldsymbol{\Omega}^*_t\right)|| = o_p(1)$.

Proof of Lemma B.1.
For part *i)*, let $\gamma \in [0,\pi]$ and denote $\vartheta_{\boldsymbol{\gamma},t-1} = \sum_{j=t}^{\infty}\omega_j(\gamma)\,\varepsilon_{t-j}$. Since $\omega_j(\gamma) = O(1/j)$, it follows that $E\left[(\vartheta_{\boldsymbol{\gamma},t})^2\right] = O\left(\sum_{j=t}^{\infty}1/j^2\right) = O\left(t^{-1}\right)$ and, therefore, $\sqrt{t}\vartheta_{\gamma,t} = O_p(1)$. Hence, $\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1} - \boldsymbol{\varepsilon}^*_{\boldsymbol{\gamma},t-1} \equiv \boldsymbol{\vartheta}_{\gamma,t} = O_p\left(t^{-1/2}\right)$. Also,

$$E||\varepsilon_t\boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}||^2 = \sum_{s=1}^{n}\sum_{j=t,l=t}^{\infty}\omega_j(\gamma_s)\,\omega_l(\gamma_s)\,E\left(\varepsilon_t^2\varepsilon_{t-j}\varepsilon_{t-l}\right),$$

where, from stationarity,

$$E\left(\varepsilon_t^2\varepsilon_{t-j}\varepsilon_{t-l}\right) = \kappa_\varepsilon(0,j,l,0) + \sigma^4\mathbb{I}_{(j=l)},$$

and, since $\kappa_\varepsilon(0,j,l,0) = o\left(\frac{1}{|j|\ |l|}\right)$ necessarily under the assumption of absolute summability, then

$$\begin{aligned}\sum_{j=t,l=t}^{\infty} \omega_j(\gamma_s)\,\omega_l(\gamma_s)\,E\left(\varepsilon_t^2\varepsilon_{t-j}\varepsilon_{t-l}\right) &= \sigma^4\sum_{j=t}^{\infty}\omega_j^2(\gamma_s) + o\left(\sum_{j=t,l=t}^{\infty}\frac{1}{j^2l^2}\right)\\ &= O\left(\sum_{j=t}^{\infty}1/j^2\right) + o\left(\sum_{j=t}^{\infty}\frac{1}{j^2}\right)o\left(\sum_{l=t}^{\infty}\frac{1}{l^2}\right)\end{aligned}$$

and therefore $E||\varepsilon_t\vartheta_{\boldsymbol{\gamma},t}||^2 = O\left(t^{-1}\right) + o\left(t^{-2}\right)$ as required. Note that, under condition A $ii.a)$, then $\kappa_\varepsilon(0,j,l,0) = 0$ and the required result simplifies trivially to $E||\varepsilon_t\vartheta_{\boldsymbol{\gamma},t}||^2 = O\left(t^{-1}\right)$. For part $ii)$, since $E\left(\varepsilon_t\varepsilon_s\varepsilon_{t-j}\varepsilon_{s-l}\right) = 0$ for all $t \neq s$ owing to the MDS property of ε_t, we have

$$\begin{aligned}E\left\|\frac{1}{T^\alpha}\sum_{t=2}^{T}\varepsilon_t\boldsymbol{\vartheta}_{\gamma,t}\right\|^2 &\leq \frac{1}{T^{2\alpha}}\sum_{t=2}^{T}E||\varepsilon_t\boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}||^2 + o(1)\\ &= \frac{1}{T^{2\alpha}}\left(\sum_{t=2}^{T}\left[O\left(t^{-1}\right)+o\left(t^{-2}\right)\right]\right) + o(1)\\ &= O\left(\frac{\log T}{T^{2\alpha}}\right) + o\left(T^{-2\alpha}\right) + o(1)\\ &= o(1)\end{aligned}$$

for any $\alpha > 0$ under Assumption $\mathcal{A}$, by using (i). From Markov's inequality, we can conclude that,

$$\left\|\frac{1}{T^\alpha}\sum_{t=2}^{T}\varepsilon_t\left(\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**} - \boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\right)\right\| = O_p\left(\sqrt{\log T}/T^\alpha\right) = o_p(1).$$

Similarly,

$$\begin{aligned}E\left\|\frac{1}{T^\alpha}\sum_{t=2}^{T}\boldsymbol{\vartheta}_{\gamma,t}\right\|^2 &\leq \frac{1}{T^{2\alpha}}\sum_{s=1}^{n}\sum_{t=2}^{T}\sum_{j=t,l=t}^{\infty}\omega_j(\gamma_s)\,\omega_l(\gamma_s)\,E\left(\varepsilon_{t-j}\varepsilon_{t-l}\right) + o(1)\\ &= \sum_{s=1}^{n}\left(\frac{1}{T^{2\alpha}}\sum_{t=2}^{T}\sum_{j=t}^{\infty}\omega_j^2(\gamma_s)\,E\left(\varepsilon_{t-j}^2\right)\right) + o(1)\\ &= O\left(\frac{\log T}{T^{2\alpha}}\right).\end{aligned}$$

For part $iii)$, first note that we can write

$$\begin{aligned}\boldsymbol{\Omega}_t^{**} - \boldsymbol{\Omega}_t^{*} &= \left(\sum_{j,l=t}^{\infty}\boldsymbol{\omega}_j\boldsymbol{\omega}_l'\varepsilon_{t-j}\varepsilon_{t-l}\right) + \left(\sum_{j=1}^{t-1}\sum_{l=t}^{\infty}\boldsymbol{\omega}_j\boldsymbol{\omega}_l'\varepsilon_{t-j}\varepsilon_{t-l}\right) + \left(\sum_{j=t}^{\infty}\sum_{l=1}^{t-1}\boldsymbol{\omega}_j\boldsymbol{\omega}_l'\varepsilon_{t-j}\varepsilon_{t-l}\right)\\ &= \mathbf{D}_{1t} + \mathbf{D}_{2t} + \mathbf{D}_{3t},\end{aligned}$$

where these terms have been defined implicitly. For the first component, note that $\mathbf{D}_{1t} = \boldsymbol{\vartheta}_{\gamma,t}\boldsymbol{\vartheta}'_{\gamma,t}$. Then, from the triangle and Cauchy-Schwarz matrix inequalities and the MDS property of ε_t it follows that

$$\begin{aligned} E\left\|\frac{1}{T^{\alpha}}\sum_{t=2}^{T}\boldsymbol{\vartheta}_{\gamma,t}\boldsymbol{\vartheta}'_{\gamma,t}\right\| &\leq \frac{1}{T^{\alpha}}\sum_{t=2}^{T}E\left\|\boldsymbol{\vartheta}_{\gamma,t}\boldsymbol{\vartheta}'_{\gamma,t}\right\| \leq \frac{1}{T^{\alpha}}\sum_{t=2}^{T}E\left\|\boldsymbol{\vartheta}_{\gamma,t}\right\|^2 \\ &\leq \sum_{i=1}^{n}\left(\frac{1}{T^{\alpha}}\sum_{t=2}^{T}\sum_{j=t}^{\infty}\omega_j^2\left(\gamma_i\right)E\left(\varepsilon_{t-j}^2\right)\right)+o\left(1\right) \\ &= O\left(\frac{\log T}{T^{\alpha}}\right), \end{aligned}$$

and, hence, $||T^{-\alpha}\sum_{t=2}^{T}\mathbf{D}_{1t}|| = o_p(1)$ for any $\alpha > 0$. Similarly, $\mathbf{D}_{2t} = \sum_{j=1}^{t-1}\omega_j\varepsilon_{t-j}\left(\sum_{l=t}^{\infty}\boldsymbol{\omega}_l\varepsilon_{t-l}\right)' = \varepsilon^{*}_{\boldsymbol{\gamma},t-1}\boldsymbol{\vartheta}'_{\gamma,t}$. Therefore, for any $\beta > 1/2$, it follows by triangle and Cauchy-Schwarz inequalities joint with the properties of the matrix norm that

$$\begin{aligned} E\left\|\frac{1}{T^{\beta}}\sum_{t=2}^{T}\mathbf{D}_{2t}\right\| &\leq \frac{1}{T^{\beta}}\sum_{t=2}^{T}E\left\|\varepsilon^{*}_{\boldsymbol{\gamma},t-1}\boldsymbol{\vartheta}'_{\gamma,t}\right\| \leq \frac{1}{T^{\beta}}\sum_{t=2}^{T}\sqrt{E\left\|\varepsilon^{*}_{\boldsymbol{\gamma},t-1}\right\|^2}\sqrt{E\left\|\boldsymbol{\vartheta}_{\gamma,t}\right\|^2} \\ &= O\left(\frac{T^{1/2}}{T^{\beta}}\right) \\ &= o_p\left(1\right) \end{aligned}$$

because $E\left\|\varepsilon^{*}_{\boldsymbol{\gamma},t-1}\right\|^2 \leq E\left\|\varepsilon^{**}_{\boldsymbol{\gamma},t-1}\right\|^2 = O(1)$ and $E\left\|\boldsymbol{\vartheta}_{\gamma,t}\right\|^2 = O(1/t)$, as discussed in (i) above. Finally, $\mathbf{D}_{3t} = \left(\sum_{j=t}^{\infty}\boldsymbol{\omega}_j\varepsilon_{t-j}\right)\left(\sum_{l=1}^{t-1}\boldsymbol{\omega}_l\varepsilon_{t-l}\right)' = \mathbf{D}'_{2t} = \boldsymbol{\vartheta}_{\gamma,t}\varepsilon'^{*}_{\boldsymbol{\gamma},t-1}$, and consequently $\left\|T^{-\beta}\sum_{t=2}^{T}\mathbf{D}_{3t}\right\| = O_p\left(\frac{T^{1/2}}{T^{\beta}}\right)$, which renders the required result. For part iv), note that $\varepsilon_t^2\left(\boldsymbol{\Omega}_t^{**}-\boldsymbol{\Omega}_t^{*}\right) = \varepsilon_t^2\left(\mathbf{D}_{1t}+\mathbf{D}_{2t}+\mathbf{D}'_{2t}\right)$, and the required result then holds as in previous lemmae. First, $\varepsilon_t^2\mathbf{D}_{1t} = \left(\varepsilon_t\boldsymbol{\vartheta}_{\gamma,t}\right)\left(\boldsymbol{\vartheta}_{\gamma,t}\varepsilon_t\right)'$, and hence, by the triangle inequality and Cauchy-Schwarz inequalities

$$E\left\|\frac{1}{T^{\alpha}}\sum_{t=2}^{T}\varepsilon_t^2\ \boldsymbol{\vartheta}_{\gamma,t}\boldsymbol{\vartheta}'_{\gamma,t}\right\| \leq \frac{1}{T}\sum_{t=2}^{T}E\left\|\varepsilon_t\boldsymbol{\vartheta}_{\gamma,t}\right\|^2 = o\left(1\right)$$

for any $\alpha > 0$ from (i). Also, $\varepsilon_t^2\mathbf{D}_{2t} = \left(\varepsilon_t\varepsilon^{*}_{\boldsymbol{\gamma},t-1}\right)\left(\boldsymbol{\vartheta}_{\gamma,t}\varepsilon_t\right)'$, so for any $\beta > 1/2$ we have

$$\begin{aligned} E\left\|\frac{1}{T^{\beta}}\sum_{t=2}^{T}\varepsilon_t^2\ \mathbf{D}_{2t}\right\| &\leq \frac{1}{T^{\beta}}\sum_{t=2}^{T}\sqrt{E\left\|\varepsilon_t\varepsilon^{*}_{\boldsymbol{\gamma},t-1}\right\|^2}\sqrt{E\left\|\varepsilon_t\boldsymbol{\vartheta}_{\gamma,t}\right\|^2} \\ &\leq \frac{1}{T^{\beta}}\sum_{t=2}^{T}\sqrt{E\left\|\varepsilon_t\varepsilon^{**}_{\boldsymbol{\gamma},t-1}\right\|^2}\sqrt{E\left\|\varepsilon_t\boldsymbol{\vartheta}_{\gamma,t}\right\|^2} \\ &= O\left(\frac{T^{1/2}}{T^{\beta}}\right) = o\left(1\right). \end{aligned}$$

Since obviously $\left\| T^{-\beta} \sum_{t=2}^{T} \varepsilon_t^2 \mathbf{D}_{2t}' \right\| = O_p\left(\frac{T^{1/2}}{T^{\beta}}\right) = o_p(1)$, this completes the proof.■

Lemma B.2. *Let* $\mathbf{\Lambda}_{\varepsilon,\gamma} = E\left(\varepsilon_t^2 \boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**} \boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}'^{**}\right)$, *with* $\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}$ *generated from* $\varepsilon_{\mathbf{d},t} = \Delta_{\boldsymbol{\gamma}}(L;\mathbf{d})\, x_t$. *Then, under Assumption* $\mathcal{A}$ *and the null hypothesis, the following result holds as* $T \to \infty$,

$$\frac{1}{\sqrt{T}}\left(\sum_{t=2}^{T} \varepsilon_{\mathbf{d},t} \boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\right) \Rightarrow \mathcal{N}\left(\mathbf{0}, \mathbf{\Lambda}_{\varepsilon,\gamma}\right),$$

with $\mathbf{\Lambda}_{\varepsilon,\gamma} = \sigma^4 \mathbf{\Gamma}_{\boldsymbol{\gamma}}$ *under Assumption* $\mathcal{A}$ *ii.a).*

Proof of Lemma B.2.

Under the null $\varepsilon_{\mathbf{d},t} = \varepsilon_t$, from which

$$\sum_{t=2}^{T} \varepsilon_{\mathbf{d},t} \boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**} = \sum_{t=2}^{T}\left(\sum_{j=1}^{\infty} \boldsymbol{\omega}_j \varepsilon_{t-j}\varepsilon_t\right) = \sum_{t=2}^{T} \mathbf{Z}_t, \text{ say,}$$

where $E\left(\mathbf{Z}_t|\mathcal{G}_{t-1}\right) = 0$, so $\{\mathbf{Z}_t, \mathcal{G}_t\}$ is a vector MDS with unconditional and conditional covariance matrices

$$\begin{aligned} E\left(\mathbf{Z}_t\mathbf{Z}_t'\right) &= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} \boldsymbol{\omega}_j \boldsymbol{\omega}_l' \, E\left(\varepsilon_t^2 \varepsilon_{t-j}\varepsilon_{t-l}\right) \equiv \mathbf{\Lambda}_{\varepsilon,\gamma}, \\ E\left(\mathbf{Z}_t\mathbf{Z}_t'|\mathcal{G}_{t-1}\right) &= \sum_{j,l=1}^{\infty} \boldsymbol{\omega}_j \boldsymbol{\omega}_l' \varepsilon_{t-j}\varepsilon_{t-l} E\left(\varepsilon_t^2|\mathcal{G}_{t-1}\right). \end{aligned}$$

It is interesting to briefly comment the conditions upon which $\mathbf{\Lambda}_{\varepsilon,\gamma}$ is well-defined. Owing to stationarity, $E\left(\varepsilon_t^2 \varepsilon_{t-j}\varepsilon_{t-l}\right) = \kappa_\varepsilon(0,j,l,0) + \sigma^4 \mathbb{I}_{(j=l)}$, and thus

$$\mathbf{\Lambda}_{\varepsilon,\gamma} = \sigma^4 \mathbf{\Gamma}_{\boldsymbol{\gamma}} + \sum_{j,l\geq 1} \boldsymbol{\omega}_j \boldsymbol{\omega}_l' \, \kappa_\varepsilon(0,j,l,0).$$

The first component is bounded and positive definite, as discussed in Appendix A. Since $\boldsymbol{\omega}_j$ is not absolute summable, the second component requires additional summability conditions making $\kappa_\varepsilon(0,j,l,0)$ negligible as $j,l \to \infty$. Under i.i.d errors, $\kappa_\varepsilon(0,j,l,0) = 0$ for all l,j, and hence $\mathbf{\Lambda}_{\varepsilon,\gamma} = \sigma^4 \mathbf{\Gamma}_{\boldsymbol{\gamma}}$ is bounded and bounded away from zero. Under the more general MDS assumption, the absolute summability of the fourth-order cumulants ensures $\mathbf{\Lambda}_{\varepsilon,\gamma} < \infty$, and as a result the asymptotic covariance matrix is characterized by the pattern of conditional heteroskedasticity. Since $\mathbf{\Lambda}_{\varepsilon,\gamma} - \sigma^4 \mathbf{\Gamma}_{\boldsymbol{\gamma}}$ is obviously semipositive definite, $\mathbf{\Lambda}_{\varepsilon,\gamma}$ is bounded and bounded away from zero.

We now prove the required result by using the central limit theory for vector MDS. For any $\boldsymbol{\lambda} \in R^n$ such that $\boldsymbol{\lambda}'\boldsymbol{\lambda} = \mathbf{1}$ define $z_t = \boldsymbol{\lambda}'\mathbf{Z}_t$. Then, we require $(\mathcal{C}1)$ $T^{-1}\sum_{t=2}^{T} z_t^2 - E\left(z_t^2\right) \xrightarrow{p} 0$, and $(\mathcal{C}2)$ $\max_{2\leq t\leq T}|z_t| > \delta \xrightarrow{p} 0$, for some $\delta > 0$, (c.f Davidson 1994, Thm

24.3). Note that $T^{-1}\sum_{t=2}^{T}\left(z_t^2 - E\left(z_t^2\right)\right) = \boldsymbol{\lambda}'\mathbf{S}_T\boldsymbol{\lambda}$, where $\mathbf{S}_T = T^{-1}\sum_{t=2}^{T}\left(\mathbf{Z}_t\mathbf{Z}_t' - \boldsymbol{\Lambda}_{\varepsilon,\gamma}\right)$ owing to the MDS property of $\mathbf{Z}_t$, and then $(\mathcal{C}1)$ is verified if $\mathbf{S}_T = o_p(1)$ by Slutsky's theorem. It is worth noting that $\sum_{l=0}^{\infty}|\omega_l(\gamma_i)\,\omega_l(\gamma_j)| < \infty$ for any $\gamma_i, \gamma_j \in [0,\pi]$ by Cauchy-Schwarz inequality, so $\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},\mathbf{t-1}}$ is defined through a $\mathcal{G}_t$-measurable transformation of a strictly stationary and ergodic process under Assumption $\mathcal{A}$. Therefore, $\mathbf{Z}_t$ is a strictly stationary and ergodic MDS (cf. White 2000, Thm. 3.35), and so is z_t.

Under Assumption $\mathcal{A}$ *ii.a)*, $T^{-1}\sum_{t=2}^{T}\left[E\left(\mathbf{Z}_t\mathbf{Z}_t'\right) - E\left(\mathbf{Z}_t\mathbf{Z}_t'|\mathcal{G}_{t-1}\right)\right] \xrightarrow{p} 0$, because $\{\mathbf{Z}_t, \mathcal{G}_t\}$ is a stationary and ergodic MDS. Furthermore, since $E\left(|\varepsilon_t|^4\right) < K < \infty$ for all t, and $E\left(\mathbf{Z}_t\mathbf{Z}_t'\right) = \sigma^4\boldsymbol{\Gamma}_{\boldsymbol{\gamma}}$, then $T^{-1}\sum_{t=2}^{T}E\left(\mathbf{Z}_t\mathbf{Z}_t'\right) \xrightarrow{p} \sigma^4\boldsymbol{\Gamma}_{\boldsymbol{\gamma}}$ from stationarity. Alternatively, under Assumption $\mathcal{A}$ *ii.b)*, for any $\gamma_i, \gamma_j \in [0,\pi]$, and the set of indices $l_h \geq 1$, $h = 1, ..., 4$, define

$$\varsigma_{ij}(l_1, l_2, l_3, l_4) = \omega_{l_1}(\gamma_i)\,\omega_{l_2}(\gamma_i)\,\omega_{l_3}(\gamma_j)\,\omega_{l_4}(\gamma_j)$$

and let $E||\mathbf{S}_T - \boldsymbol{\Lambda}_{\varepsilon,\gamma}||^2 = \sum_{i,j=1}^{n}\mathcal{E}_{ij,T}$, whose characteristic element is given by

$$\begin{aligned}\mathcal{E}_{ij,T} &= E\left(\frac{1}{T}\sum_{t=2}^{T}\varepsilon_t^2\varepsilon^{**}_{\gamma_i,t-1}\varepsilon^{**}_{\gamma_j,t-1} - \left[\boldsymbol{\Lambda}_{\varepsilon,\gamma}\right]_{ij}\right)^2 \\ &= T^{-1}\sum_{l_1,...,l_4=1}^{\infty}\varsigma_{ij}(l_1,...,l_4)\left\{T^{-1}\sum_{t=2}^{T}\sum_{s=2}^{T}Cov\left(\varepsilon_{t-l_1}\varepsilon_{t-l_2}\varepsilon_t^2,\varepsilon_{s-l_3}\varepsilon_{s-l_4}\varepsilon_s^2\right)\right\} + o(1).\end{aligned}$$

The covariances on the right-hand side do not depend on any of the elements of $\boldsymbol{\gamma}$. Furthermore, under the assumption of stationarity, they can be written as the sum of products of cumulants of ε_t of order eight and lower (cf. Brillinger 1981, Thm. 2.3.2), which eventually rule the asymptotic behavior of $\mathcal{E}_{ij,T}$. First, we examine the case $i = j = 1$ related to $\gamma_1 = 0$. Under the restriction of absolute summability, $T|\mathcal{E}_{11,T}|$ is uniformly bounded by

$$\sum_{\tau=-\infty}^{\infty}\sum_{l_1,...,l_4=1}^{\infty}|\varsigma_{11}(l_1,...,l_4)|\;|\kappa_\varepsilon(0, l_1 - l_4, l_1, l_1, \tau - l_3 + l_1, \tau - l_4 + l_1, \tau + l_1, \tau + l_1)|$$

with $\tau \equiv t - s$; see Gonçalves and Kilian (2007) and Proposition 2 in Demetrescu *et al.* (2007). By Lemma 10 in the latter paper, and noting that $\varsigma_{11}(l_1,...,l_4) = O\left(\frac{1}{l_1\times...\times l_4}\right)$, this term can be shown to be uniformly bounded as well. Then, for the generic term $\mathcal{E}_{ij,T}$, $i, j \geq 1$, and noting that $|\omega_j(\gamma)|$ is uniformly bounded in $[0,\pi]$ by $2/j$, it follows for any pair $\gamma_i, \gamma_j \in [0,\pi]$ that

$$|\varsigma_{ij}(l_1,...,l_4)| \leq \prod_{h=1}^{4}|2l_h^{-1}| \;\leq\; 8\,|\varsigma_{11}(l_1,...,l_4)|,$$

from which obviously $T\,|\mathcal{E}_{ij,T}| \;\leq 8T\,|\mathcal{E}_{11,T}| \;< K < \infty$, independently of T or the particular frequencies involved. Consequently, $E||\mathbf{S}_T - \boldsymbol{\Lambda}_{\varepsilon,\gamma}||^2 = O\left(T^{-1}\right) = o(1)$ and $T^{-1}\sum_{t=2}^{T}\varepsilon_t^2\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},\mathbf{t-1}}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},\mathbf{t-1}} \xrightarrow{ms} \boldsymbol{\Lambda}_{\varepsilon,\gamma}$. Since mean-square convergence implies convergence in probability, $(\mathcal{C}1)$ holds under Assumption $\mathcal{A}$ as required. At this point it is worth recalling that $\left\|T^{-1}\sum_{t=2}^{T}\varepsilon_t^2\left(\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1} - \boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\right)\right\| =$

$o_p(1)$ from Lemma B.1 *iii*), so it follows by the Asymptotic Equivalence Lemma [AEL] (cf. White 2000, Lemma 4.7) and under the null hypothesis that

$$T^{-1}\sum_{t=2}^{T}\varepsilon_{\mathbf{d},t}^{2}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},\mathbf{t-1}}^{*}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},\mathbf{t-1}}\xrightarrow{p}\boldsymbol{\Lambda}_{\varepsilon,\gamma}.$$

To address $(\mathcal{C}2)$ recall that, under Assumption $\mathcal{A}$, $\{\mathbf{Z}_t, z_t\}$ is strictly stationary and ergodic, and uniformly bounded and bounded away from zero under the L_2-norms, so the Lindeberg condition in $(\mathcal{C}2)$ is trivially satisfied (cf. Davidson 2000, Thm. 6.2.3). Therefore, the Central Limit Theorem (CLT) for MDS joint with the Cramér-Wold device (cf. Davidson 1994, Thm. 25.6) allows us to conclude under the null hypothesis and as $T\to\infty$ that $T^{-1/2}\sum_{t=2}^{T}\varepsilon_t^2\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\Rightarrow\mathcal{N}\left(0,\boldsymbol{\Lambda}_{\varepsilon,\boldsymbol{\gamma}}\right)$. To complete the proof, recall from Lemma B.1 *ii*) $\left\|T^{-1/2}\sum_{t=2}^{T}\varepsilon_t^2\boldsymbol{\vartheta}_{\gamma,t}\boldsymbol{\vartheta}'_{\gamma,t}\right\|=o_p(1)$, so by the AEL it follows that,

$$T^{-1/2}\sum_{t=2}^{T}\varepsilon_t^2\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\Rightarrow\mathcal{N}\left(0,\boldsymbol{\Lambda}_{\varepsilon,\boldsymbol{\gamma}}\right)$$

as required. This completes the proof.■

Lemma B.3. *Define the k-th order autocovariance* $E\left(\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1-k}\right)=\boldsymbol{\Lambda}_{\varepsilon,\gamma}(k)$, $k>0$, *and let* $\widehat{e}_t$ *be the estimated residuals from the auxiliary regression (20). Then, the following results hold under Assumption* $\mathcal{A}$, *the null hypothesis, and* $T\to\infty$:

i) $\sum_{k=0}^{\infty}\boldsymbol{\Lambda}_{\varepsilon,\gamma}^{p}(k)<\infty$ *for* $p\geq 1$;
ii) $T^{-1}\sum_{t=2}^{T}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},t-1}\xrightarrow{p}\sigma^2\boldsymbol{\Gamma}_{\boldsymbol{\gamma}}$;
iii) $T^{-1}\sum_{t=2}^{T}\widehat{e}_t^2\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},t-1}\xrightarrow{p}\boldsymbol{\Lambda}_{\varepsilon,\gamma}$, *with* $\boldsymbol{\Lambda}_{\varepsilon,\gamma}\equiv E\left(\varepsilon_t^2\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1}\right)$.

Proof of Lemma B.3.
In *i*), the asymptotic k-th order autocovariance matrix, $k\geq 0$, is given by

$$E\left(\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1-k},\right)=\sum_{j,l=1}^{\infty}\boldsymbol{\omega}_j\boldsymbol{\omega}'_l E\left(\varepsilon_{t-j}\varepsilon_{t-k-l}\right)=\sigma^2\sum_{j=1}^{\infty}\boldsymbol{\omega}_j\boldsymbol{\omega}'_{j+k}\equiv\boldsymbol{\Lambda}_{\varepsilon,\gamma}(k)<\infty$$

with $\boldsymbol{\Lambda}_{\varepsilon,\gamma}(0)=\boldsymbol{\Lambda}_{\varepsilon,\boldsymbol{\gamma}}$. More specifically,

$$\boldsymbol{\Lambda}_{\varepsilon,\gamma}(k)=o\left(\sum_{j=1}^{\infty}\frac{1}{j(j+k)}\right)=o\left(\frac{1}{k}\left(\sum_{j=1}^{\infty}\frac{1}{j}-\frac{1}{j+k}\right)\right)=o\left(\frac{\log k}{k}\right),$$

and, as a result, $\left\{\boldsymbol{\Lambda}_{\varepsilon,\gamma}^{p}(k)\right\}_{k=0}^{\infty}$ is summable for any $p\geq 1$. For part *ii*), let again $\boldsymbol{\Omega}_t^{**}=\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1}$ and $\boldsymbol{\Omega}_t^{*}=\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},t-1}$, with $\overline{\boldsymbol{\Omega}}_T^{**}$ and $\overline{\boldsymbol{\Omega}}_T^{*}$ being their respective sample means.

Clearly, $E\left(\overline{\mathbf{\Omega}}_T^{**}\right) = \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}$, whereas $\overline{\mathbf{\Omega}}_T^{*}$ is asymptotically unbiased, since

$$\begin{aligned} E\left(\overline{\mathbf{\Omega}}_T^{*}\right) &= \sigma^2\sum_{j=1}^{T}\boldsymbol{\omega}_j\boldsymbol{\omega}_j' - \sigma^2T^{-1}\sum_{j=2}^{T} j\left[\boldsymbol{\omega}_j\boldsymbol{\omega}_j'\right] + \sigma^2T^{-1}\sum_{j=2}^{T}\boldsymbol{\omega}_j\boldsymbol{\omega}_j' \\ &= \sigma^2\sum_{j=1}^{T}\boldsymbol{\omega}_j\boldsymbol{\omega}_j' - o(1) + O\left(T^{-1}\right) \\ &\to \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}. \end{aligned}$$

We can show that $\overline{\mathbf{\Omega}}_T^{**} \overset{ms}{\to} \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}$ using a similar approach as in Lemma B.2., from which $\overline{\mathbf{\Omega}}_T^{*} \overset{p}{\to} \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}$ by Lemma B.1 *iii*) and the AEL. In particular, note that we can write

$$\begin{aligned} TE\left(\left[\frac{1}{T}\sum_{t=2}^{T}\mathbf{\Omega}_t^{**} - \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}\right]_{ij}\right)^2 &= \sum_{l_1,\ldots,l_4=1}^{\infty}\varsigma_{ij}\left(l_1,\ldots,l_4\right) \\ &\quad\times\frac{1}{T}\sum_{t=2}^{T}\sum_{s=2}^{T}Cov\left(\left[\varepsilon_{t-l_1}\varepsilon_{t-l_2}\right],\left[\varepsilon_{s-l_3}\varepsilon_{s-l_4}\right]\right) + o(1). \end{aligned}$$

Following Lemma A.2 in Gonçalves and Kilian (2004) and Lemma 8 in Demetrescu *et al.* (2007), this term is uniformly bounded by $\mathfrak{B}_{ij} + 2\sum_{k=-\infty}^{\infty}\left[\mathbf{\Lambda}_{\varepsilon,\gamma}^2(k)\right]_{ij}$, with $\mathbf{\Lambda}_{\varepsilon,\gamma}(k)$ defined in (i) and

$$\mathfrak{B}_{ij} = \sum_{t=-\infty}^{\infty}\sum_{l_1,\ldots,l_4=0}^{\infty}\left|\varsigma_{ij}\left(l_1,\ldots,l_4\right)\right|\ \left|\kappa_\varepsilon\left(0,l_2-l_1,t+l_3-l_1,t+l_4-l_1\right)\right|.$$

Since $\left|\varsigma_{ij}\left(l_1,\ldots,l_4\right)\right| \leq 8\ \left|\varsigma_{11}\left(l_1,\ldots,l_4\right)\right|$ and $\sum_{k=-\infty}^{\infty}\mathbf{\Lambda}_{\varepsilon,\gamma}^2(k) < \infty$ from stationarity and according to (i), then for any pair $\gamma_i,\gamma_j \in [0,\pi]$, $\mathfrak{B}_{ij} + 2\sum_{k=-\infty}^{\infty}\left[\mathbf{\Lambda}_{\varepsilon,\gamma}^2(k)\right]_{ij} < \infty$ as a corollary of Lemma 8 in Demetrescu *et al.* (2007). Hence, $E||\overline{\mathbf{\Omega}}_T^{**} - \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}||^2 = O\left(T^{-1}\right)$ and therefore $\overline{\mathbf{\Omega}}_T^{**} \overset{ms}{\to} \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}$. But since $||T^{-1}\sum_{t=2}^{T}\left(\mathbf{\Omega}_t^{**} - \mathbf{\Omega}_t^{*}\right)|| = o_p(1)$ from Lemma B.1 *iv*), the AEL allows us to conclude for the observable sample mean process

$$T^{-1}\sum_{t=2}^{T}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{\prime *} \overset{p}{\to} \sigma^2\mathbf{\Gamma}_{\boldsymbol{\gamma}},$$

as required, with convergence in probability being implied by the stronger convergence in the mean square sense.

In *iii*), the null hypothesis implies $\boldsymbol{\phi} = \mathbf{0}$ and $e_t = \varepsilon_t$ in the auxiliary regression, thereby $\widehat{e}_t^2 - \varepsilon_t^2 = \left(\sum_{s=1}^{n}\phi_{s,T}\varepsilon_{\gamma_s,t-1}^{**}\right)^2 = \left(\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{\prime **}\boldsymbol{\phi}_T\right)\left(\boldsymbol{\phi}_T'\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\right)$. Hence,

$$\begin{aligned} \left\|\frac{1}{T}\sum_{t=2}^{T}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\left(\widehat{e}_t^2 - \varepsilon_t^2\right)\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{\prime **}\right\| &= \left\|\frac{1}{T}\sum_{t=2}^{T}\mathbf{\Omega}_t^{**}\boldsymbol{\phi}_T\boldsymbol{\phi}_T'\mathbf{\Omega}_t^{**}\right\| \leq \frac{1}{T}\sum_{t=2}^{T}||\mathbf{\Omega}_t^{**}\boldsymbol{\phi}_T\boldsymbol{\phi}_T'\mathbf{\Omega}_t^{**}|| \\ &\leq \frac{1}{T}\sum_{t=2}^{T}||\mathbf{\Omega}_t^{**}||\ ||\boldsymbol{\phi}_T\boldsymbol{\phi}_T'||\ ||\mathbf{\Omega}_t^{**}|| \end{aligned}$$

by the triangle inequality, first, and the Cauchy-Schwarz inequality matrix, finally. The estimated parameter vector $\boldsymbol{\phi}_T$ is $\sqrt{T}$-consistent (see proof in Theorem 4.1 below), so $\|\boldsymbol{\phi}_T\boldsymbol{\phi}'_T\| = O_p\left(T^{-1}\right)$. Since

$$E\left\|\boldsymbol{\Omega}_t^{**}\right\|^2 \leq \sum_{i,j=1}^{n}\sum_{l_1,...,l_4=1}^{\infty}\left|\varsigma_{ij}\left(l_1,...,l_4\right)\right|\left|E\left(\varepsilon_{t-l_1}\varepsilon_{t-l_2}\varepsilon_{t-l_3}\varepsilon_{t-l_4}\right)\right|$$

is uniformly bounded from Assumption $\mathcal{A}$ it follows that

$$\left\|\frac{1}{T}\sum_{t=2}^{T}\left(\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{\prime**}\right)\left(\widehat{e}_t^2-\varepsilon_t^2\right)\right\| = O_p\left(\frac{1}{T}\sum_{t=2}^{T}O\left(T^{-1}\right)\right) = O_p\left(T^{-1}\right)$$

as T diverges. Finally, as Lemma B.1 *iv*), we can readily show that $\left\|T^{-1}\sum_{t=2}^{T}\left(\boldsymbol{\vartheta}_{\gamma,t-1}\boldsymbol{\vartheta}'_{\gamma,t-1}\right)\left(\widehat{e}_t^2-\varepsilon_t^2\right)\right\| = o_p(1)$, so the AEL renders the required result.■

Proof of Theorem 3.1.

The proof of Theorem 3.1 is now obvious in view of the results in Lemmas B.1.-B.2., and holds straightforwardly by the Continuos Mapping Theorem (CMT). In particular

$$\begin{aligned} LM_T &= \left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_{\mathbf{d},t}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\right)'\left[\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{\mathbf{d},t}^2\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{\prime*}\right]^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_{\mathbf{d},t}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{*}\right) \\ &= \left(\mathbf{A}'_T\right)\left[\mathbf{B}_T^{-1}\right]\left(\mathbf{A}_T\right), \text{ say.} \end{aligned}$$

Under the null hypothesis, $\varepsilon_{\mathbf{d},t} = \varepsilon_t$, so under Assumption $\mathcal{A}$, $\mathbf{A}_T \Rightarrow \mathcal{N}\left(0, \boldsymbol{\Lambda}_{\varepsilon,\boldsymbol{\gamma}}\right)$ and $\mathbf{B}_T \xrightarrow{p} \boldsymbol{\Lambda}_{\varepsilon,\boldsymbol{\gamma}}$ as $T \to \infty$ according to Lemma B.1 *i*) and *iv*), Lemma B.2, and the AEL. The required convergence then follows by the CMT from which $LM_T \Rightarrow \mathbf{N}'_n\mathbf{N}_n$, where $\mathbf{N}_n$ is a n-dimensional standard normal distribution and, hence, $LM_T \Rightarrow \chi^2_{(n)}$.■

Proof of Theorem 4.1.

Let $\boldsymbol{\phi}_T^{**}$ be the OLS estimator in $\varepsilon_{\mathbf{d},t} = \phi_1\varepsilon_{\gamma_1,t-1}^{**} + \phi_2\varepsilon_{\gamma_2,t-1}^{**} + ... + \phi_n\varepsilon_{\gamma_n,t-1}^{**} + e_t$. Since under the null hypothesis $\boldsymbol{\phi} = \mathbf{0}$ and $e_t = \varepsilon_{\mathbf{d},t} = \varepsilon_t$, then

$$\begin{aligned} \sqrt{T}\boldsymbol{\phi}_T^{**} &= \left(\frac{1}{T}\sum_{t=2}^{T}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{\prime**}\right)^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_{\mathbf{t}}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\right) \\ &= \left(\mathbf{C}_T\right)^{-1}\left(\mathbf{A}_T\right), \end{aligned}$$

where $\mathbf{A}_T$ is defined as in the proof of Theorem 3.1 above. Hence, under Assumption $\mathcal{A}$, and as $T \to \infty$, it follows from Lemmas B.1-B.3 and the CMT that

$$\sqrt{T}\boldsymbol{\phi}_T^{**} \Rightarrow \mathcal{N}\left(\mathbf{0}, \left(\frac{1}{\sigma^4}\right)\boldsymbol{\Gamma}_{\boldsymbol{\gamma}}^{-1}\boldsymbol{\Lambda}_{\varepsilon,\gamma}\boldsymbol{\Gamma}_{\boldsymbol{\gamma}}^{-1}\right),$$

so that ϕ_T^{**} is $\sqrt{T}$-consistent and asymptotically normal. If the errors are i.i.d, $\mathbf{\Lambda}_{\varepsilon,\gamma}=\sigma^4\mathbf{\Gamma}_{\boldsymbol{\gamma}}$ and the asymptotic covariance matrix reduces to $\mathbf{V}_{\boldsymbol{\gamma}}=\mathbf{\Gamma}_{\boldsymbol{\gamma}}^{-1}$. Similarly, from Lemmas B.1 and B.2 and the CMT, it follows that

$$\sqrt{T}\boldsymbol{\phi}_T \Rightarrow \mathcal{N}\left(0,\mathbf{V}_{\boldsymbol{\gamma}}\right)$$

as required. This completes the proof.■

Proof of Theorem 4.2.
The proof of the convergence of the regression based test statistic $\Upsilon_W^{(n)}$ is immediate from the asymptotic normality in Theorem 4.1 and holds as a corollary. Note that

$$\Upsilon_W^{(n)}=\left(\sqrt{T}\boldsymbol{\phi}_T\right)'\left[\mathbf{V}_{\boldsymbol{\gamma},T}\right]^{-1}\left(\sqrt{T}\boldsymbol{\phi}_T\right),$$

where $\mathbf{V}_{\boldsymbol{\gamma},T}$ can be estimated by using either $\widehat{e}_t$ or $\varepsilon_{\mathbf{d},t}$ in the sample estimate of $\mathbf{\Lambda}_{\varepsilon,\boldsymbol{\gamma}}$, as shown in Lemma B.3 *iii*), whereas $E\left(\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{**}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma},t-1}^{\prime **}\right)$ can be estimated consistently as either $\overline{\mathbf{\Omega}}_T^*$ or $\left(\hat{\sigma}_T^2\mathbf{\Gamma}_{\boldsymbol{\gamma}}\right)$, with $\hat{\sigma}_T^2$ being the sample variance of either $\varepsilon_{\mathbf{d},t}$ or $\widehat{e}_t$, and $\mathbf{\Gamma}_{\boldsymbol{\gamma}}$ determined numerically or by means of the close-form representations in Appendix A. From Theorem 3.1, $\Upsilon_W^{(n)}$ behaves asymptotically as a Gaussian quadratic form, and then the CMT ensures $\Upsilon_W^{(n)}\Rightarrow\chi^2_{(n)}$.■

Corollaries.
Consider the auxiliary regression in Corollary 4.1 when the auxiliary regression has only $1\leq m<n$ variables,

$$\varepsilon_{\mathbf{d},t}=\phi_1\varepsilon^*_{\gamma_1,t-1}+\phi_2\varepsilon^*_{\gamma_2,t-1}+...+\phi_m\varepsilon^*_{\gamma_m,t-1}+e_{t,m}$$

but $\varepsilon_{\mathbf{d},t}=\Delta_{\boldsymbol{\gamma}}\left(L;\mathbf{d}\right)x_t$, $\mathbf{d}\in R^n$. For simplicity of notation, but with no loss of generality, assume the regressors correspond to the first m frequencies in $\boldsymbol{\gamma}$ and define $\boldsymbol{\gamma}_m=\left(\gamma_1,...,\gamma_m\right)'$. Under $\mathrm{H}_0:\boldsymbol{\theta}=\mathbf{0}$, $\varepsilon_{\mathbf{d},t}=\varepsilon_t$ and both the dependent variable and the regressors preserve the asymptotic properties discussed in the main test. Consequently,

$$\sqrt{T}\boldsymbol{\phi}_{T,m}\Rightarrow\mathcal{N}\left(\mathbf{0},\left(\frac{1}{\sigma^4}\right)\mathbf{\Gamma}_{\gamma_m}^{-1}\mathbf{\Lambda}_{\varepsilon,\gamma_m}\mathbf{\Gamma}_{\gamma_m}^{-1}\right)$$

with $\boldsymbol{\phi}_{T,m}=\left(\phi_{1,T},...,\phi_{m,T}\right)'$, $\mathbf{\Gamma}_{\gamma_m}=\sum_{j=1}^{\infty}\boldsymbol{\omega}_j\left(\boldsymbol{\gamma}_m\right)\boldsymbol{\omega}_j'\left(\boldsymbol{\gamma}_m\right)$, and $\mathbf{\Lambda}_{\varepsilon,m}$ corresponding to the upper-corner sub-matrix of $\mathbf{\Lambda}_{\varepsilon,\gamma}$. The asymptotic distribution of the Wald-type test for $\mathrm{H}_0=\phi_1=...=\phi_m=0$ is now $\chi^2_{(m)}$.
For Corollary 4.2, notice that the score of the log-likelihood function when $\boldsymbol{\theta}=\theta\mathbf{1}_n$, with $\mathbf{1}_n$

being a vector of ones in R^n, is given by

$$
\begin{aligned}
\left.\frac{\partial \mathcal{L}(\boldsymbol{\delta}, \sigma^2|\mathbf{x}_T)}{\partial \theta}\right|_{\mathrm{H}_0:\theta=0} &= -\frac{1}{\sigma^2}\sum_{t=1}^{T}\varepsilon_t\left(\log\left[1-L\right]+\sum_{i=2}^{n-1}\log\left[\xi_{\gamma_i}(L;1)\right]+\log\left[1+L\right]\right)\varepsilon_t \\
&= \frac{1}{\sigma^2}\sum_{t=1}^{T}\varepsilon_t\sum_{s=1}^{n}\left(\sum_{j=1}^{\infty}\omega_j(\gamma_s)\,\varepsilon_{t-j}\right) \\
&\equiv \frac{1}{\sigma^2}\sum_{t=1}^{T}\varepsilon_t\left(\sum_{s=1}^{n}\varepsilon^{**}_{\gamma_s,t-1}\right)
\end{aligned}
$$

which suggest that $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$ can be tested by analyzing the statistical significance of the $\bar{\phi}$ parameter in the auxiliary regression $\varepsilon_{\mathbf{d},t} = \bar{\phi}\left(\sum_{s=1}^{n}\varepsilon^{*}_{\gamma_s,t-1}\right) + u_t$. Since $\sum_{s=1}^{n}\varepsilon^{*}_{\gamma_s,t-1} = \mathbf{1}_n'\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}$ is a linear transformation of the regressors in the basic auxiliary regression, we have that

$$
\bar{\phi}_T = \left(\mathbf{1}_n'\bar{\boldsymbol{\Omega}}^{*}_T\mathbf{1}_n\right)^{-1}\left(\mathbf{1}_n'\left[\varepsilon_t\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\right]\right)
$$

and, hence, it follows from Theorem 3.2 and the CMT that $\sqrt{T}\bar{\phi}_T \Rightarrow \mathcal{N}\left(0, \mathbf{1}_n'\mathbf{V}_{\boldsymbol{\gamma}}\mathbf{1}_n\right)$ as $T \to \infty$.■

Proofs for weakly correlated errors

Lemma B.4. *Let $\{b_j\}_{j\geq 0}$ be the coefficients in the Wold representation, $\varepsilon_t = \sum_{j=0}^{\infty} b_j v_{t-j}$ under Assumption $\mathcal{B}$. Let $\varphi_j(\gamma)$ be the j-th element in the serial convolution of $\{\omega_{j+1}(\gamma)\}_{j\geq 0}$ and $\{b_j\}_{j\geq 0}$ for any $\gamma \in [0,\pi]$. Then, $\varphi_j(\gamma) = \omega_1(\gamma)$, if j=0, and $\varphi_j(\gamma)$ is $O(\omega_j(\gamma))$ otherwise.*

Proof of Lemma B.4.
Recall that, for all $\gamma \in [0,\pi]$, $|\omega_j(\gamma)| \leq 2/j$, and hence $\omega_j(\gamma) = O(1/j)$. The serial convolution of $\{b_j\}_{j\geq 0}$ and $\{\omega_{j+1}\}_{j\geq 0}$ determines coefficients as a function of the γ frequency which are given by

$$
\varphi_j(\gamma) = \sum_{k=0}^{j} b_k\omega_{j-k+1}(\gamma),
$$

where $\varphi_j(\gamma) \leq |\varphi_j(\gamma)| \leq 2\sum_{k=0}^{j}\frac{j}{j-k+1}|b_k|$, with $\left(\frac{j}{j-k+1}\right) \leq k$ for all $1 \leq k \leq j$, so $\varphi_j(\gamma) \leq |b_0|\left(\frac{2j}{j+1}\right) + 2\sum_{j=1}^{k} j|b_k|$. Since for any stationary AR(p) model $\sum_{j=1}^{k} j|b_k| < \infty$, the coefficient $|\varphi_j(\gamma)|$ is bounded by a constant as $j \to \infty$, and hence $\varphi_j(\gamma) = O(1/j)$, which leads us to the desired result.

As a result, $\{\varphi_j(\gamma)\}$ belongs to the same space of squared-summable coefficient series as $\{\omega_j(\gamma)\}$ does, so the results discussed under MDS errors follow under Assumption $\mathcal{B}$ in most cases by simply modifying the limit variances. Also, note that since γ is taken from $[0, \pi]$, this lemma trivially generalizes the results in Demetrescu *et al.* (2007), discussed for $\gamma = 0$, to any other frequency.■

Lemma B.5. *Under Assumption $\mathcal{B}$, the asymptotic and truncated processes under the null hypothesis are now given by* $\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1} = \sum_{j=0}^{\infty} \boldsymbol{\varphi}_j v_{t-j-1}$, $\boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1} = \sum_{j=0}^{t-1} \boldsymbol{\varphi}_j v_{t-j-1}$, *with* $\boldsymbol{\varphi}_j \equiv (\varphi_j(\gamma_1), ..., \varphi_j(\gamma_n))'$, and $\{\varphi_j(\cdot)\}_{j\geq 0}$ *given in Lemma B.4. Then, as T is allowed to diverge, Lemma B.1 still holds under Assumption $\mathcal{B}$ with trivial modifications, i.e.,: i)* $\boldsymbol{\vartheta}_{\boldsymbol{\gamma},t} = O_p\left(t^{-1/2}\right)$, and $E||v_t \boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}|| = O\left(t^{-1}\right) + o\left(t^{-2}\right)$, *ii)* $||T^{-\alpha} \sum_{t=p+1}^{T} v_t \boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}|| = o_p(1)$ and $||T^{-\alpha} \sum_{t=p+1}^{T} \boldsymbol{\vartheta}_{\boldsymbol{\gamma},t}|| = o_p(1)$, *iii)* $||T^{-\beta} \sum_{t=p+1}^{T} \left(\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1} - \boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},t-1}\right)|| = o_p(1)$, *iv)* $||T^{-\beta} \sum_{t=p+1}^{T} v_t^2 \left(\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}'^{**}_{\boldsymbol{\gamma},t-1} - \boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}'^{*}_{\boldsymbol{\gamma},t-1}\right)|| = o_p(1)$, *for any* $\alpha > 0$, $\beta > 1/2$.

Proof of Lemma B.5. It holds directly from Lemma B.4 and Lemma B.1.

Lemma B.6. *Let* $\mathbf{X}_{tp} = (\varepsilon_{\mathbf{d},t-1}, ..., \varepsilon_{\mathbf{d},t-p})'$ *be the p-dimensional vector with the lagged values of the dependent variable, and define the $n + p$ dimensional vectors* $\mathbf{X}^{*}_{tp} = \left(\boldsymbol{\varepsilon}^{*\prime}_{\boldsymbol{\gamma},t-1}, \mathbf{X}'_{tp}\right)'$, $\mathbf{X}^{**}_{tp} = \left(\boldsymbol{\varepsilon}^{**\prime}_{\boldsymbol{\gamma},t-1}, \mathbf{X}'_{tp}\right)'$. *Define* $\boldsymbol{\Omega}^{**}_p = E\left(\mathbf{X}^{**}_{tp}\mathbf{X}'^{**}_{tp}\right)$, *and let* $\overline{\boldsymbol{\Omega}}^{*}_p = T^{-1}\sum_{t=2}^{T} \mathbf{X}^{*}_{tp}\mathbf{X}'^{*}_{tp}$. *Then, i)* $\boldsymbol{\Omega}^{**}_p$ *is bounded and bounded away from zero, and ii)* $||\overline{\boldsymbol{\Omega}}^{*}_p - \boldsymbol{\Omega}^{**}_p|| = o_p(1)$.

Proof of Lemma B.6.
For part *i*), note that $\boldsymbol{\Omega}^{**}_p$ can be partitioned as

$$\boldsymbol{\Omega}^{**}_p \equiv \begin{pmatrix} [\boldsymbol{\Sigma}_{\varepsilon\gamma}]_{\mathrm{n}\times\mathrm{n}} & [\boldsymbol{\Sigma}'_{\varepsilon X}]_{\mathrm{n}\times\mathrm{p}} \\ [\boldsymbol{\Sigma}_{\varepsilon X}]_{\mathrm{p}\times\mathrm{n}} & [\boldsymbol{\Sigma}_{X}]_{\mathrm{p}\times\mathrm{p}} \end{pmatrix},$$

where $\boldsymbol{\Sigma}_{\varepsilon\gamma} = \sigma^4 \sum_{j=1}^{\infty} \boldsymbol{\varphi}_j \boldsymbol{\varphi}'_j$ is positive definite and bounded because $\{\varphi_j(\gamma)\}$ is square-summable. Similarly, $\boldsymbol{\Sigma}_X = \sigma^2 \sum_{j=1}^{\infty} \mathbf{b}_j \mathbf{b}'_j$, with $\mathbf{b}_j = (b_{j-1}, ..., b_{j-p})'$ and $b_l = 0$ for all $l < 0$, is finite and positive definite owing to absolute summability of the coefficients in the Wald's representation of any stationary AR(p) process. From the Cauchy-Schwarz inequality, $||\boldsymbol{\Sigma}_{\varepsilon X}|| \leq ||\boldsymbol{\Sigma}_{\varepsilon\gamma}||^{1/2} ||\boldsymbol{\Sigma}_X||^{1/2} < \infty$, from which $||\boldsymbol{\Omega}^{**}_p|| < \infty$. Finally, $\boldsymbol{\Omega}^{**}_p$ is singular if and only if the elements of $\mathbf{X}^{**}_{tp}$ are linearly dependent, which obviously is not the case, so $\det(\boldsymbol{\Omega}^{**}_p) > \delta > 0$. Part *ii*) holds if ($a$) $||\bar{\boldsymbol{\Sigma}}^{*}_{\varepsilon\gamma} - \boldsymbol{\Sigma}_{\varepsilon,\gamma}|| = o_p(1)$, ($b$) $||\bar{\boldsymbol{\Sigma}}^{*}_{X} - \boldsymbol{\Sigma}_X|| = o_p(1)$, and ($c$) $||\bar{\boldsymbol{\Sigma}}^{*}_{\varepsilon X} - \boldsymbol{\Sigma}_{\varepsilon X}|| = o_p(1)$, given the respective sample estimators, *e.g.*, $\bar{\boldsymbol{\Sigma}}^{*}_{\varepsilon\gamma} = (T-p)^{-1} \sum_{t=p+1}^{T} \boldsymbol{\varepsilon}^{*}_{\boldsymbol{\gamma},t-1}\boldsymbol{\varepsilon}^{*\prime}_{\boldsymbol{\gamma},t-1}$. The proof of ($a$) follows from B.4 and B.5 and identically as in Lemma B.3. The proof of (b) follows as in Theorem 2.2 in Gonçalves and Kilian (2004). Finally, for part (c), define

$$\begin{aligned} A_{1tT} &= \sum_{j=0}^{\infty} b_j \varphi_{j+i-1}(\gamma_k) \left[v^2_{t-j-i} - \sigma^2\right], \\ A_{2tT} &= \sum_{l=0}^{T} \sum_{\substack{j=0 \\ j\neq l-i+1}}^{T} b_l \varphi_j(\gamma_k)\, v_{t-j-1} v_{t-l-i} \end{aligned}$$

and let $E||\bar{\mathbf{\Sigma}}_{\varepsilon X}^{**} - \mathbf{\Sigma}_{\varepsilon X}||^2 = \sum_i^p \sum_k^n \sum_{t,s=p+1}^T E\left[\left(A_{1tT} + A_{2tT}\right)\left(A_{1sT} + A_{2sT}\right)\right]$. Notice that

$$\begin{aligned} T^{-2} \sum_{t=p+1}^{T} \sum_{s=p+1}^{T} E\left(A_{1tT} A_{1sT}\right) &= \frac{1}{T} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} b_j \varphi_j\left(\gamma\right) b_l \varphi_l\left(\gamma_k\right) \times \\ & \quad \left\{ \frac{1}{T} \sum_{t=p+1}^{T} \sum_{s=p+1}^{T} Cov\left(v_{t-j-i-1}^2, v_{s-l-i-1}^2\right) \right\} \end{aligned}$$

by setting $b_j = \varphi_l\left(\gamma_k\right) = 0$ for all $j, l < 0$. Under the restriction of stationarity and absolutely summable cumulants, the term in curly brackets is uniformly bounded in t, s and T for any $1 \leq i \leq p < \infty$. Hence, given some constant $K < \infty$, it follows by the Cauchy-Schwarz inequality that

$$\begin{aligned} T^{-2} \sum_{t=p+1}^{T} \sum_{s=p+1}^{T} E\left(A_{1t} A_{1s}\right) &\leq \frac{K}{T} \left(\sum_{j=0}^{\infty} b_j \varphi_j\left(\gamma\right) \right)^2 \leq \frac{K}{T} \left(\sum_{j=0}^{\infty} b_j^2 \right) \left(\sum_{j=0}^{\infty} \varphi_{j+i}^2\left(\gamma\right) \right) \\ &= O\left(T^{-1}\right). \end{aligned}$$

Similarly, under Assumption $\mathcal{B}$, we can show that the remaining term,

$$T^{-2} \sum_{t=p+1}^{T} \sum_{s=p+1}^{T} E\left(A_{2tT} A_{2sT}\right) + T^{-2} \sum_{t=p+1}^{T} \sum_{s=p+1}^{T} E\left(A_{1sT} A_{2tT} + A_{1tT} A_{2sT}\right) = O\left(T^{-1}\right)$$

from which $||\bar{\mathbf{\Sigma}}_{\varepsilon X}^{**} - \mathbf{\Sigma}_{\varepsilon X}|| = O_p\left(T^{-1/2}\right) = o_p\left(1\right)$ by Markov's inequality. Finally, as in Lemma B.2, we can show

$$||\bar{\mathbf{\Sigma}}_{\varepsilon X}^{**} - \bar{\mathbf{\Sigma}}_{\varepsilon X}^{*}|| = O_p\left(T^{-1} \sum_{t=p+1}^{T} \left(\mathbf{\Omega}_t^{**} - \mathbf{\Omega}_t^{*}\right) \right) = O_p\left(T^{-1/2}\right) = o_p\left(1\right),$$

and then the AEL renders the required result. This completes the proof. ■

Lemma B.7. *Let $\mathbf{\Lambda}_p = E\left(v_t^2 \mathbf{X}_{tp}^{**} \mathbf{X}_{tp}^{\prime **}\right)$ be defined through the partition*

$$\begin{pmatrix} \left[\mathbf{\Lambda}_{\varepsilon\gamma}^{b}\right]_{\mathrm{n}\times\mathrm{n}} & \left[\mathbf{\Lambda}_{\varepsilon X}^{\prime}\right]_{\mathrm{n}\times\mathrm{p}} \\ \left[\mathbf{\Lambda}_{\varepsilon X}\right]_{\mathrm{p}\times\mathrm{n}} & \left[\mathbf{\Lambda}_{X}\right]_{\mathrm{p}\times\mathrm{p}} \end{pmatrix}$$

and let e_{tp} and $\widehat{e}_{tp}$ be the residuals, and the estimated residuals, respectively, from the augmented auxiliary regression. Then, under then null hypothesis, Assumption $\mathcal{B}$, and as $T \rightarrow \infty$:

i) $\mathbf{\Lambda}_p < \infty$, *and* $det(\mathbf{\Lambda}_p) > \delta > 0$;
ii) $T^{-1/2} \sum_{t=p+1}^{T} e_{tp} \mathbf{X}_{tp}^{*} \Rightarrow \mathcal{N}\left(0, \mathbf{\Lambda}_p\right)$;
iii) $T^{-1} \sum_{t=p+1}^{T} \widehat{e}_{tp}^{2} \left(\mathbf{X}_{tp}^{*} \mathbf{X}_{tp}^{\prime *}\right) \xrightarrow{p} \mathbf{\Lambda}_p$.

Proof of Lemma B.7.

For part i), $\mathbf{\Lambda}^b_{\varepsilon\gamma} = \mathbf{\Sigma}_{\varepsilon\gamma} + \sum_{j,l\geq 1}\left[\boldsymbol{\varphi}_j\boldsymbol{\varphi}_l'\right]\kappa_v(0,j,l,0)$, with $\mathbf{\Sigma}_{\varepsilon\gamma} = \sigma^4\sum_{j=1}^{\infty}\boldsymbol{\varphi}_j\boldsymbol{\varphi}_j'$ defined in Lemma B.6. From Lemma B.4 and Assumption $\mathcal{B}$, the same statistical considerations as in Lemma B.2 apply on $\mathbf{\Lambda}^b_{\varepsilon\gamma}$, and as a result this a finite, positive definite covariance matrix. Similarly, we can show as in Theorem 2.2 in Gonçalves and Kilian (2007) that $\mathbf{\Lambda}_X < \infty$, whereas from the Cauchy-Schwarz inequality $\mathbf{\Lambda}_{\varepsilon X} < \infty$, from which $\mathbf{\Lambda}_p < \infty$. As in B.6, $\mathbf{\Lambda}_p$ is invertible, and so $\det(\mathbf{\Lambda}_p) > \delta > 0$. For part ii), under the null hypothesis $e_{tp} = v_t$, so given $\mathbf{v}_{t-1} = (v_{t-1}, ..., v_{t-p})'$, we have

$$\sum_{t=p+1}^{T} e_{tp}\begin{pmatrix}\boldsymbol{\varepsilon}^{**}_{\boldsymbol{\gamma},t-1}\\ \mathbf{X}_{tp}\end{pmatrix} = \sum_{t=p+1}^{T}\left[\sum_{j=0}^{\infty}\begin{pmatrix}\boldsymbol{\varphi}_j v_{t-j-1}v_t\\ b_j\mathbf{v}_{t-j-1}v_t\end{pmatrix}\right] = \sum_{t=p+1}^{T}\begin{pmatrix}\mathbf{Z}_{\varepsilon t}\\ \mathbf{Z}_{Xt}\end{pmatrix}, \text{ say.}$$

Clearly, $\{\mathbf{Z}_{\varepsilon t}, \mathcal{F}_t\}$ and $\{\mathbf{Z}_{Xt}, \mathcal{F}_t\}$, $\mathcal{F}_t = \sigma(v_j : j \leq t)$, are squared-integrable MDS under Assumption $\mathcal{B}$, with $E(\mathbf{Z}_{\varepsilon t}\mathbf{Z}_{\varepsilon t}') = \mathbf{\Lambda}^b_{\varepsilon\gamma}$, $E(\mathbf{Z}_{Xt}\mathbf{Z}_{Xt}') = \mathbf{\Lambda}_X$, and $E(\mathbf{Z}_{\varepsilon t}\mathbf{Z}_{Xt}') = \mathbf{\Lambda}_{\varepsilon X}'$. We can use the CLT for MDS as in Lemma B.2 to show asymptotic normality of the normalized sums of $(\mathbf{Z}_{\varepsilon t}', \mathbf{Z}_{Xt}')'$. In particular, note that $(\mathcal{C}1)$ holds if a) $||\bar{\mathbf{\Lambda}}^b_{\varepsilon\gamma,T} - \mathbf{\Lambda}^b_{\varepsilon\gamma}|| = o_p(1)$, b) $||\bar{\mathbf{\Lambda}}_{X,T} - \mathbf{\Lambda}_X|| = o_p(1)$, and c) $||\bar{\mathbf{\Lambda}}_{\varepsilon X,T} - \mathbf{\Lambda}_{\varepsilon X}|| = o_p(1)$, where again the first terms denote the sample estimates based on the filtered process. The proof of a) follows along the same lines as in Lemma B.2 owing to Lemma B.4. The proof of b) follows as in Theorem 3.1 in Gonçalves and Killian (2004). To check c), note that for $1 \leq i \leq p$, and $1 \leq k \leq n$, the characteristic element of $TE||\bar{\mathbf{\Lambda}}_{\varepsilon X,T} - \mathbf{\Lambda}_{\varepsilon X}||^2$ can be written as $T^{-1}\sum_{t=p+1}^{T}\sum_{s=p+1}^{T} Cov\left(\varepsilon_{\mathbf{d},t-i}\varepsilon_{\gamma_k,t-1}v_t^2, \varepsilon_{\mathbf{d},s-i}\varepsilon_{\gamma_k,s-1}v_s^2\right)$, *i.e.*,

$$\begin{aligned}&T^{-1}\sum_{l_1,\ldots,l_4=-\infty}^{\infty} b_{l_1}b_{l_3}\varphi_{l_2}(\gamma_k)\varphi_{l_4}(\gamma_k)\\ &\times\sum_{t=p+1}^{T}\sum_{s=p+1}^{T} Cov\left(v_{t-i-l_1-1}v_{t-l_2-1}v_t^2, v_{s-i-l_3-1}v_{s-l_4-1}v_s^2\right)\end{aligned}$$

with $b_l = \varphi_l(\gamma_k) = 0$ for all $l < 0$. First, consider the zero-frequency case for which $k = 1$. As discussed in Proposition 2 in Demetrescu *et al.* (2007), this term is uniformly bounded by a constant that does not depend on t, s, T or i. Then, for any $1 \leq k \leq n$ and all $1 \leq i \leq p$, note that

$$|b_{l_1}b_{l_3}\varphi_{l_2}(\gamma_k)\varphi_{l_4}(\gamma_k)| \;\leq 4\;|b_{l_1}b_{l_3}\varphi_{l_2}(0)\varphi_{l_4}(0)|$$

and as a result it is immediate to show that $E||\bar{\mathbf{\Lambda}}_{\varepsilon X,T} - \mathbf{\Lambda}_{\varepsilon X}||^2 = O(T^{-1}) = o_p(1)$, thus implying

$$\frac{1}{T-p}\sum_{t=p+1}^{T} v_t^2\left(\mathbf{X}^{**}_{tp}\mathbf{X}'^{**}_{tp}\right) \overset{ms}{\to} \mathbf{\Lambda}_p$$

as required. Finally note that, from Lemma B.4, $(\mathbf{Z}_{\varepsilon t}', \mathbf{Z}_{Xt}')'$ is defined by an $\mathcal{F}_t$-measurable function on $\{v_t\}$, so it is a strictly stationary and ergodic MDS (cf. White 2000, Thm. 3.35).

Furthermore, from (i) in this lemma, the process is bounded and bounded away from zero under the L_2-norms, so $(\mathcal{C}2)$ holds trivially. Hence, as $T \to \infty$, Assumption $\mathcal{B}$, and under the null hypothesis $T^{-1/2} \sum_{t=p+1}^{T} e_{tp} \mathbf{X}_{tp}^{**} \Rightarrow \mathcal{N}(0, \mathbf{\Lambda}_p)$. Finally, since $||T^{-1/2} \sum_{t=p+1}^{T} v_t \left(\mathbf{X}_{tp}^{**} - \mathbf{X}_{tp}^{*}\right)|| = o_p(1)$ from Lemma B.5, it follows by the AEL that

$$T^{-1/2} \sum_{t=p+1}^{T} e_{tp} \mathbf{X}_{tp}^{*} \Rightarrow \mathcal{N}(0, \mathbf{\Lambda}_p)$$

as required. For part $iii)$, consider $a_{i,T}$ the LS estimate of the i-th autoregressive coefficient. Then,

$$v_t - \widehat{e}_{tp} = \sum_{i=1}^{p} (a_{i,T} - a_i) \varepsilon_{\mathbf{d},t-i} + \sum_{k=1}^{n} \phi_{k,T} \varepsilon_{\gamma_k,t-1}^{*} = O_p\left(T^{-1/2}\right)$$

owing to $\sqrt{T}$-consistency (see Theorem 4.3 below). Therefore, $v_t^2 - \widehat{e}_{tp}^2 = (v_t - \widehat{e}_{tp})(v_t + \widehat{e}_{tp}) = O_p\left(T^{-1/2}\right) + O_p\left(T^{-1}\right)$, and hence

$$\begin{aligned} \left\| T^{-1} \sum_{t=p+1}^{T} \left(v_t^2 - \widehat{e}_{tp}^2\right) \mathbf{X}_{tp}^{**} \mathbf{X}_{tp}^{\prime **} \right\| &\leq \frac{1}{T} \sum_{t=p+1}^{T} \left|v_t^2 - \widehat{e}_{tp}^2\right| \left\| \mathbf{X}_{tp}^{**} \mathbf{X}_{tp}^{\prime **} \right\| \\ &= O_p\left(T^{-1/2}\right) \\ &= o_p(1) \end{aligned}$$

which together with (ii) above implies that $\frac{1}{T-p} \sum_{t=p+1}^{T} \widehat{e}_{tp}^2 \left(\mathbf{X}_{tp}^{**} \mathbf{X}_{tp}^{\prime **}\right) \overset{ms}{\to} \mathbf{\Lambda}_p$ by the AEL. But since

$$\begin{aligned} \left\| T^{-1} \sum_{t=p+1}^{T} \left(v_t^2 - \widehat{e}_{tp}^2\right) \left(\mathbf{X}_{tp}^{**} \mathbf{X}_{tp}^{\prime **} - \mathbf{X}_{tp}^{*} \mathbf{X}_{tp}^{\prime *}\right) \right\| &= O_p\left(T^{-1} \sum_{t=p+1}^{T} \left(v_t^2 - \widehat{e}_{tp}^2\right) \left(\mathbf{\Omega}_t^{**} - \mathbf{\Omega}_t^{*}\right) \right) \\ &= O_p\left(T^{-1} \sum_{t=p+1}^{T} O_p\left(T^{-1/2}\right) O_p\left(1/\sqrt{t}\right) \right) \\ &= O_p\left(T^{-1/2}\right) \\ &= o_p(1) \end{aligned}$$

by using Cauchy-Schwarz inequality, it follows from the AEL that

$$\frac{1}{T-p} \sum_{t=p+1}^{T} \widehat{e}_{tp}^2 \mathbf{X}_{tp}^{*} \mathbf{X}_{tp}^{\prime *} = \mathbf{\Lambda}_p + o_p(1)$$

as T is allowed to diverge. This completes the proof.■

Proof of Theorem 4.3.

The proof of Theorem 4.3 is immediate in view of the previous results. Let $\boldsymbol{\beta}_T^{**}$ and $\boldsymbol{\beta}_T$ be the OLS estimations in the corresponding augmented auxiliary regressions $\varepsilon_{\mathbf{d},t} = \mathbf{X}_{tp}^{\prime **}\boldsymbol{\beta}^{**} + e_{tp}$, and $\varepsilon_{\mathbf{d},t} = \mathbf{X}_{tp}^{\prime *}\boldsymbol{\beta} + e_{tp}$, respectively. Since

$$\sqrt{T}\left(\boldsymbol{\beta}_T - \boldsymbol{\mu}_0\right) = \left(\frac{1}{T}\sum_{t=p+1}^{T} \mathbf{X}_{tp}^{*}\mathbf{X}_{tp}^{*\prime}\right)^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=p+1}^{T} e_{tp}\mathbf{X}_{tp}^{*}\right)$$

then according to lemmae B.4-B.7 and the CMT, it follows under Assumption $\mathcal{B}$, the null hypothesis, and as $T \to \infty$, that $\sqrt{T}\left(\boldsymbol{\beta}_T^{**} - \boldsymbol{\mu}_0\right)$ and $\sqrt{T}\left(\boldsymbol{\beta}_T - \boldsymbol{\mu}_0\right)$ are asymptotically equivalent, with

$$\sqrt{T}\left(\boldsymbol{\beta}_T - \boldsymbol{\mu}_0\right) \Rightarrow \mathcal{N}\left(\mathbf{0},\ \left(\boldsymbol{\Omega}_p^{**}\right)^{-1}\boldsymbol{\Lambda}_p\left(\boldsymbol{\Omega}_p^{**}\right)^{-1}\right).$$

■

Proof of Theorem 4.4.

Given normality in the estimated coefficients, Theorem 4.4 holds as a corollary of Theorem 4.3. Let $\mathbf{R}$ be an $n \times (n+p)$ matrix such $[\mathbf{R}]_{ij} = 1$ for all $i = j$ and zero otherwise. Consider the regression-based test statistic on the estimates of the augmented auxiliary regression, i.e.,

$$\Upsilon_{Wp}^{(n)} = \left[\sqrt{T}\left(\mathbf{R}\boldsymbol{\beta}_T\right)\right]^{\prime}\left[\mathbf{R}\widehat{\mathbf{V}}_{\mathbf{T}}\mathbf{R}^{\prime}\right]^{-1}\left[\sqrt{T}\mathbf{R}\left(\boldsymbol{\beta}_T\right)\right]$$

where $\widehat{\mathbf{V}}_{\mathbf{T}}$ is the sample counterpart of the asymptotic covariance matrix of $\boldsymbol{\beta}_T$, *i.e.*,

$$\widehat{\mathbf{V}}_{\mathbf{T}} = \left(\frac{1}{T}\sum_{t=p+1}^{T}\mathbf{X}_{tp}^{*}\mathbf{X}_{tp}^{*\prime}\right)^{-1}\left(\frac{1}{T}\sum_{t=p+1}^{T}\widehat{e}_{tp}^{2}\mathbf{X}_{tp}^{*}\mathbf{X}_{tp}^{*\prime}\right)\left(\frac{1}{T}\sum_{t=p+1}^{T}\mathbf{X}_{tp}^{*}\mathbf{X}_{tp}^{*\prime}\right)^{-1}$$

where the inclusion of the squared estimated residuals, $\widehat{e}_{tp}^{2}$, is intended to provided robustness against (conditional) heteroskedastic patterns of unknown form. Given the previous lemmae and the CMT, it follows readily that

$$\sqrt{T}\left(\mathbf{R}\hat{\boldsymbol{\beta}}_T\right) = \sqrt{T}\boldsymbol{\phi}_T \Rightarrow \mathcal{N}\left(0, \mathbf{R}\left[\left(\boldsymbol{\Omega}_{\mathbf{p}}^{**}\right)^{-\mathbf{1}}\boldsymbol{\Lambda}_{\mathbf{p}}\left(\boldsymbol{\Omega}_{\mathbf{p}}^{**}\right)^{-\mathbf{1}}\right]\mathbf{R}^{\prime}\right)$$

under the null hypothesis and as the sample length diverges, $\Upsilon_{Wp}^{(n)}$ converges to the distribution of a Gaussian quadratic form and therefore $\Upsilon_{Wp}^{(n)} \Rightarrow \chi_{(n)}^{2}$.■

Corollaries.

Corollary 4.3 holds from asymptotic normality in Theorem 4.3 owing to the fact that $\boldsymbol{\Lambda}_p \propto \boldsymbol{\Omega}_p^{**}$, see Theorems 4.32 and 4.37 and comments in White (2000). Similarly, Corollary 4.4 holds as Corollary 4.2.

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1 Tables and Figures

Table 1: Empirical rejection frequencies when the DGP is the Simple GARMA model $(1 - 2\cos\gamma_s L + L^2)^{1+\theta} x_t = \varepsilon_t, \quad \varepsilon_t \sim iidn(0, 1)$.

				θ			
γ_s	-.3	-.2	-.1	0	.1	.2	.3
				T=100			
$\frac{\pi}{10}$	.999	.984	.540	**.052**	.584	.981	.999
$\frac{2\pi}{10}$	.999	.933	.401	**.054**	.445	.927	.998
$\frac{3\pi}{10}$	.988	.810	.302	**.056**	.329	.832	.982
$\frac{4\pi}{10}$	.946	.689	.232	**.049**	.267	.721	.946
$\frac{5\pi}{10}$	.929	.630	.210	**.050**	.248	.686	.932
$\frac{6\pi}{10}$	.955	.683	.236	**.051**	.269	.730	.947
$\frac{7\pi}{10}$	.985	.826	.311	**.045**	.331	.836	.985
$\frac{8\pi}{10}$	.998	.929	.425	**.051**	.452	.933	.998
$\frac{9\pi}{10}$	.999	.982	.536	**.050**	.585	.984	.999
				T=250			
$\frac{\pi}{10}$	.999	.999	.924	**.043**	.921	.999	.999
$\frac{2\pi}{10}$	.999	.999	.818	**.057**	.814	.999	.999
$\frac{3\pi}{10}$	.999	.997	.653	**.050**	.686	.995	.999
$\frac{4\pi}{10}$	.999	.979	.516	**.052**	.563	.980	.999
$\frac{5\pi}{10}$	.999	.971	.468	**.051**	.545	.968	.999
$\frac{6\pi}{10}$	.999	.980	.520	**.051**	.571	.978	.999
$\frac{7\pi}{10}$	.999	.998	.664	**.045**	.682	.994	.999
$\frac{8\pi}{10}$	.999	1.00	.811	**.050**	.816	.999	.999
$\frac{9\pi}{10}$	.999	.999	.918	**.045**	.913	.999	.999

Note: Empirical size is in bold.

Table 2: Empirical rejection frequencies when the DGP is the 2-factor GARMA model $(1-2\cos\gamma_1 L+L^2)^{1+\theta_1}(1-2\cos\gamma_2 L+L^2)^{1+\theta_2}x_t=\varepsilon_t,\ \varepsilon_t\sim iidn(0,1)$ and T=100

Test on θ_1, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.999	.999	.999	.999	.999	.999	.999
-.2	.975	.981	.987	.992	.999	.999	.999
-.1	.411	.452	.508	.608	.707	.827	.924
.0	**.143**	**.098**	**.073**	**.053**	**.062**	**.103**	**.244**
.1	.868	.812	.751	.633	.488	.282	.123
.2	.999	.998	.995	.988	.975	.936	.817
.3	.999	.999	.999	.999	.999	.999	.994

2-lags Augmented Test on θ_1, $\gamma_2=\frac{\pi}{2}$

θ_1	-.3	-.2	-.1	0	.1	.2	.3
-.3	.636	.714	.815	.883	.950	.978	.993
-.2	.361	.393	.460	.514	.604	.698	.762
-.1	.182	.165	.159	.152	.145	.156	.165
.0	**.088**	**.069**	**.052**	**.047**	**.048**	**.053**	**.068**
.1	.062	.059	.061	.098	.158	.269	.400
.2	.070	.054	.076	.140	.286	.489	.695
.3	.094	.066	.068	.125	.267	.511	.756

Test on θ_2, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.772	.430	.114	**.071**	.364	.774	.955
-.2	.756	.375	.088	**.072**	.398	.799	.965
-.1	.814	.408	.107	**.061**	.358	.778	.954
.0	.923	.625	.202	**.046**	.253	.660	.929
.1	.994	.912	.597	**.187**	.113	.444	.814
.2	.999	.997	.953	**.707**	.308	.213	.497
.3	.999	.999	.999	**.976**	.835	.502	.318

2-lags Augmented Test on θ_2, $\gamma_2=\frac{\pi}{2}$

θ_1	-.3	-.2	-.1	0	.1	.2	.3
-.3	.584	.370	.157	**.074**	.082	.174	.313
-.2	.646	.362	.176	**.073**	.079	.178	.311
-.1	.638	.361	.154	**.057**	.076	.190	.344
.0	.601	.312	.116	**.045**	.097	.249	.417
.1	.554	.260	.083	**.046**	.129	.322	.528
.2	.539	.232	.064	**.043**	.180	.424	.661
.3	.610	.284	.087	**.049**	.191	.475	.710

Joint Restricted Test, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.999	.999	0.997	.959	.741	.362	.247
-.2	.996	.992	.963	.834	.512	.220	.237
-.1	.793	.731	.611	.398	.179	.098	.290
.0	.126	.102	.082	**.047**	.067	.205	.480
.1	.631	.590	.583	.574	.625	.730	.853
.2	.987	.985	.982	.981	.982	.988	.993
.3	.999	.999	.999	.999	.999	.999	.999

Joint Unrestricted Test, $\gamma_2=\frac{\pi}{2}$

θ_1	-.3	-.2	-.1	0	.1	.2	.3
-.3	.999	.999	.999	.999	.999	.999	.999
-.2	.994	.978	.974	.977	.990	.999	.999
-.1	.892	.684	.502	.487	.693	.911	.985
.0	.857	.510	.161	**.049**	.205	.592	.893
.1	.988	.913	.741	.556	.535	.718	.898
.2	.999	.999	.992	.980	.974	.981	.991
.3	.999	.999	.999	.999	.999	.999	.999

Note: Empirical size is in bold.

Table 3: Empirical rejection frequencies when the DGP is the 2-factor GARMA model with ARMA errors: $(1-2\cos\gamma_1 L+L^2)^{1+\theta_1}(1-2\cos\gamma_2 L+L^2)^{1+\theta_2}x_t=\varepsilon_t,\ (1-0.5L)\varepsilon_t=(1+0.5L)v_t,\ v_t\sim iidn(0,1)$

T=100

Test on θ_1

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.078	.105	.162	.215	.285	.373	.427
-.2	.032	.046	.063	.090	.145	.173	.214
-.1	.034	.031	.045	.044	.061	.079	.093
.0	**.050**	**.049**	**.046**	**.045**	**.042**	**.046**	**.053**
.1	.077	.069	.068	.061	.059	.060	.057
.2	.097	.095	.100	.090	.092	.093	.093
.3	.127	.127	.125	.134	.132	.134	.145

Test on θ_2

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.160	.070	.035	**.051**	.096	.197	.276
-.2	.181	.088	.048	**.049**	.086	.162	.243
-.1	.199	.098	.056	**.042**	.065	.124	.182
.0	.183	.113	.060	**.043**	.053	.087	.134
.1	.150	.100	.062	**.047**	.050	.060	.085
.2	.097	.071	.055	**.039**	.043	.053	.065
.3	.063	.059	.043	**.042**	.044	.046	.054

Joint Restricted Test

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.300	.233	.148	.088	.067	.099	.142
-.2	.131	.120	.089	.059	.051	.072	.127
-.1	.063	.056	.055	.045	.041	.057	.096
.0	.047	.043	.046	**.043**	.049	.062	.080
.1	.065	.059	.063	.060	.061	.075	.086
.2	.093	.087	.092	.094	.092	.104	.113
.3	.126	.127	.123	.136	.127	.130	.139

Joint Unrestricted Test

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.204	.142	.122	.141	.202	.315	.381
-.2	.156	.097	.058	.069	.115	.160	.228
-.1	.137	.075	.046	.039	.058	.094	.138
.0	.121	.076	.046	**.037**	.044	.063	.090
.1	.113	.079	.058	.053	.053	.062	.075
.2	.103	.077	.073	.061	.068	.075	.085
.3	.105	.094	.085	.096	.091	.100	.105

T=500

Test on θ_1

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.808	.841	.870	.891	.905	.936	.949
-.2	.523	.533	.509	.505	.498	.493	.499
-.1	.242	.218	.187	.159	.128	.110	.097
.0	**.092**	**.078**	**.064**	**.051**	**.048**	**.041**	**.053**
.1	.091	.093	.096	.109	.129	.153	.193
.2	.241	.256	.276	.292	.328	.360	.408
.3	.435	.446	.469	.490	.524	.542	.573

Test on θ_2

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.750	.344	.078	**.058**	.300	.639	.825
-.2	.645	.306	.077	**.053**	.222	.508	.737
-.1	.488	.242	.071	**.051**	.162	.383	.586
.0	.295	.134	.067	**.052**	.113	.279	.437
.1	.126	.078	.045	**.050**	.095	.190	.290
.2	.052	.048	.045	**.046**	.066	.110	.167
.3	.049	.044	.042	**.042**	.055	.063	.081

Joint Restricted Test

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	.0	.1	.2	.3
-.3	.992	.955	.691	.225	.082	.316	.626
-.2	.897	.794	.525	.190	.071	.228	.534
-.1	.492	.389	.230	.093	.049	.179	.424
.0	.150	.113	.073	**.048**	.067	.175	.388
.1	.087	.090	.089	.115	.159	.258	.405
.2	.239	.255	.272	.294	.345	.401	.475
.3	.437	.448	.471	.493	.530	.543	.578

Joint Unrestricted Test

$\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	.0	.1	.2	.3
-.3	.981	.926	.834	.802	.862	.949	.979
-.2	.871	.680	.463	.386	.480	.653	.815
-.1	.570	.354	.170	.117	.177	.333	.518
.0	.264	.128	.064	**.053**	.092	.206	.360
.1	.126	.095	.075	.092	.134	.222	.338
.2	.192	.205	.215	.227	.272	.341	.405
.3	.371	.367	.394	.411	.446	.475	.511

Note: Empirical size is in bold. All tests are augmented using Schwert's rule.

Table 4: Empirical rejection frequencies when the DGP is the 2-factor GARMA model with AR errors: $(1-2\cos\gamma_1 L+L^2)^{1+\theta_1}(1-2\cos\gamma_2 L+L^2)^{1+\theta_2}x_t=\varepsilon_t,\ (1-0.5L)\varepsilon_t=v_t,\ v_t\sim iidn(0,1)$

T=100

Test on θ_1, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.160	.200	.249	.302	.348	.420	.479
-.2	.052	.077	.095	.133	.180	.240	.289
-.1	.028	.035	.044	.056	.089	.126	.176
0	**.052**	**.045**	**.044**	**.048**	**.057**	**.080**	**.096**
.1	.090	.072	.059	.055	.055	.068	.086
.2	.129	.108	.087	.078	.075	.075	.083
.3	.155	.139	.130	.113	.100	.107	.101

Test on θ_1, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.229	.102	.045	**.040**	.104	.194	.305
-.2	.263	.119	.055	**.044**	.093	.192	.291
-.1	.320	.150	.065	**.047**	.082	.178	.275
0	.340	.158	.070	**.047**	.080	.167	.250
.1	.319	.160	.070	**.044**	.075	.148	.249
.2	.253	.138	.064	**.042**	.077	.143	.240
.3	.174	.094	.052	**.041**	.079	.153	.237

Joint Restricted Test, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	0.1	.2	.3
-.3	.479	.336	.202	.108	.077	.104	.179
-.2	.250	.201	.139	.078	.061	.094	.160
-.1	.103	.087	.076	.049	.059	.093	.145
0	.056	.047	.050	**.041**	.050	.079	.132
.1	.062	.056	.054	.057	.062	.083	.122
.2	.104	.088	.083	.082	.088	.097	.135
.3	.139	.130	.123	.124	.126	.137	.151

Joint Unrestricted Test, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.334	.234	.191	.194	.255	.352	.467
-.2	.229	.138	.092	.089	.140	.232	.319
-.1	.224	.108	.055	.045	.082	.156	.249
0	.232	.101	.052	**.038**	.058	.127	.190
.1	.226	.118	.059	.046	.061	.115	.187
.2	.210	.124	.071	.057	.076	.128	.203
.3	.181	.118	.094	.083	.110	.164	.219

T=500

Test on θ_1, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.572	.705	.817	.905	.966	.986	.996
-.2	.219	.301	.390	.513	.623	.735	.815
-.1	.074	.090	.117	.137	.173	.220	.270
0	**.045**	**.052**	**.047**	**.046**	**.045**	**.054**	**.046**
.1	.118	.113	.117	.128	.134	.137	.144
.2	.285	.293	.301	.301	.343	.366	.385
.3	.465	.487	.498	.523	.549	.569	.618

Test on θ_2, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.948	.533	.093	**.069**	.420	.789	.937
-.2	.967	.661	.155	**.049**	.328	.721	.904
-.1	.962	.674	.188	**.047**	.258	.632	.864
0	.928	.628	.200	**.048**	.201	.550	.795
.1	.840	.516	.162	**.048**	.164	.462	.687
.2	.608	.335	.125	**.048**	.132	.353	.559
.3	.371	.195	.079	**.046**	.095	.234	.386

Joint Restricted Test, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.999	.966	.631	.153	.142	.527	.814
-.2	.967	.889	.564	.158	.108	.458	.766
-.1	.641	.538	.298	.093	.088	.409	.731
0	.205	.155	.088	**.044**	.118	.394	.706
.1	.094	.087	.090	.123	.225	.460	.693
.2	.232	.251	.269	.295	.398	.548	.710
.3	.438	.463	.481	.521	.574	.654	.750

Joint Unrestricted Test, $\gamma_2=\frac{\pi}{2}$

$\gamma_1=0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.995	.945	.833	.829	.934	.984	.998
-.2	.987	.812	.487	.397	.601	.853	.954
-.1	.947	.656	.236	.106	.268	.604	.834
0	.874	.513	.156	**.044**	.154	.464	.712
.1	.761	.437	.179	.101	.200	.453	.672
.2	.643	.431	.295	.247	.342	.527	.677
.3	.582	.496	.441	.441	.505	.600	.712

Note: Empirical size is in bold. All tests are augmented using Schwert's rule.

Table 5: Empirical rejection frequencies when the DGP is the 2-factor GARMA model with AR errors: $(1-2\cos\gamma_1 L + L^2)^{1+\theta_1}(1-2\cos\gamma_2 L + L^2)^{1+\theta_2} x_t = \varepsilon_t,\ (1-0.9L)\varepsilon_t = v_t,\ v_t \sim iidn(0,1)$

T=100

Test on θ_1

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.017	.025	.027	.034	.056	.071	.097
-.2	.025	.023	.031	.033	.042	.065	.082
-.1	.041	.035	.034	.036	.044	.064	.080
0	**.051**	**.048**	**.050**	**.052**	**.055**	**.067**	**.082**
.1	.069	.071	.071	.070	.064	.071	.086
.2	.079	.076	.071	.081	.090	.078	.082
.3	.077	.078	.079	.076	.076	.077	.078

Test on θ_2

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.433	.217	.098	**.049**	.073	.141	.220
-.2	.443	.241	.107	**.048**	.071	.131	.230
-.1	.401	.198	.092	**.048**	.076	.151	.239
0	.315	.155	.064	**.040**	.080	.163	.267
.1	.205	.109	.058	**.047**	.088	.182	.285
.2	.131	.080	.043	**.053**	.095	.185	.273
.3	.082	.057	.051	**.051**	.085	.140	.217

Joint Restricted Test

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.085	.080	.062	.043	.040	.067	.114
-.2	.044	.042	.046	.037	.032	.056	.100
-.1	.037	.036	.040	.040	.041	.051	.078
0	.047	.042	.046	**.051**	.049	.065	.079
.1	.062	.068	.070	.072	.068	.082	.097
.2	.076	.075	.073	.084	.095	.087	.100
.3	.073	.077	.078	.078	.078	.086	.087

Joint Unrestricted Test

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.290	.143	.064	.034	.056	.105	.168
-.2	.297	.149	.070	.035	.047	.094	.167
-.1	.272	.125	.062	.039	.050	.107	.173
0	.230	.107	.056	**.043**	.065	.123	.208
.1	.157	.097	.058	.056	.081	.142	.226
.2	.120	.080	.055	.069	.094	.149	.219
.3	.081	.068	.056	.059	.078	.118	.170

T=500

Test on θ_1

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.243	.281	.326	.367	.427	.477	.536
-.2	.144	.161	.157	.181	.183	.204	.231
-.1	.068	.075	.065	.071	.069	.071	.062
0	**.053**	**.054**	**.052**	**.049**	**.046**	**.045**	**.041**
.1	.098	.110	.086	.086	.086	.086	.083
.2	.136	.129	.128	.120	.114	.101	.117
.3	.106	.100	.092	.090	.081	.083	.082

Test on θ_2

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.989	.828	.343	**.056**	.198	.575	.830
-.2	.972	.739	.287	**.057**	.189	.539	.791
-.1	.891	.591	.214	**.054**	.168	.485	.736
0	.719	.409	.143	**.054**	.157	.406	.644
.1	.469	.229	.090	**.041**	.114	.291	.488
.2	.214	.130	.068	**.042**	.082	.181	.291
.3	.081	.066	.044	**.049**	.058	.091	.154

Joint Restricted Test

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.913	.840	.667	.295	.083	.247	.587
-.2	.547	.474	.348	.184	.063	.173	.477
-.1	.172	.164	.118	.072	.0400	.111	.351
0	.063	.062	.055	**.051**	.053	.106	.248
.1	.088	.103	.080	.085	.100	.130	.197
.2	.133	.126	.123	.123	.119	.121	.161
.3	.105	.099	.092	.090	.081	.085	.088

Joint Unrestricted Test

$\gamma_2 = \frac{\pi}{2}$

$\gamma_1 = 0.15$	-.3	-.2	-.1	0	.1	.2	.3
-.3	.995	.886	.549	.287	.391	.679	.866
-.2	.964	.738	.340	.148	.233	.524	.764
-.1	.843	.511	.185	.062	.138	.405	.656
0	.620	.319	.111	**.051**	.123	.320	.549
.1	.408	.217	.103	.066	.122	.258	.432
.2	.232	.150	.111	.090	.117	.184	.286
.3	.113	.091	.077	.076	.075	.102	.144

Note: Empirical size is in bold. All tests are augmented using Schwert's rule.

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